Introductory Partial Differential Equations

Midterm Exam

March 21, 2013

1. Let $g = C^1(\mathbb{R})$. Solve the partial differential equation

$$\begin{aligned} (x+t)(u_x + u_t) = 0 \qquad & \text{in } \mathbb{R} \times (0,\infty)\,, \\ u = g \qquad & \text{on } \mathbb{R} \times \{t=0\}\,. \end{aligned}$$

(10)

2. Suppose that a radial function $u = u(|x|)$ is harmonic on $B(0,1) \subset \mathbb{R}^n$. Show that $u \equiv \text{const.}$ (10)

3. Suppose that $U \subset \mathbb{R}^n$ is open, connected, and bounded with smooth boundary. Suppose further that $u \in C^2(\overline{U})$ solves the *Neumann problem* for the Poisson equation

$$\begin{aligned} -\Delta u = f \qquad & \text{in } U\,, \\ \nu \cdot Du = g \qquad & \text{on } \partial U \end{aligned}$$

for some $f \in C(\overline{U})$ and $g \in C(\partial U)$, where ν denotes the outer unit normal on ∂U. Show that any other solution differs from u by only a constant. (10)

4. Suppose that $u \in C_1^2(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$ solves the heat equation

$$u_t - \Delta u = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty)$$

and is a Gaussian at the initial time, i.e.,

$$u(x,0) = a\,\mathrm{e}^{-b\,|x|^2}$$

with some $a \in \mathbb{R}$ and $b > 0$. Prove that u remains Gaussian for all times $t > 0$. (10)

5. Recall that the solution to the heat equation

$$\begin{aligned} u_t - \Delta u = 0 \quad & \text{in } \mathbb{R}^n \times (0,\infty)\,, \\ u = g \quad & \text{on } \mathbb{R}^n \times \{t=0\} \end{aligned}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy\,,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|z|^2}{4t}}\,.$$

Assume that g is continuous and compactly supported. Show that there exists a $C > 0$ such that

$$|Du(x,t)| \leq \frac{C}{\sqrt{t}}\, \|g\|_{L^\infty}\,.$$

(10)

6. Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary. Let $b \in C^1(\overline{U})$ satisfy

$$\operatorname{div} b \equiv D \cdot b = 0 \qquad \text{in } U\,,$$
$$\nu \cdot b = 0 \qquad \text{on } \partial U\,.$$

Further, suppose that $u \in C^1(\overline{U} \times [0,T])$ solves the transport equation

$$u_t + b \cdot Du = 0 \qquad \text{in } U\,.$$

(a) Show that

$$M = \int_U u\, dx$$

is constant in time.

(b) In a modeling scenario, u could describe the concentration of a certain substance in the container U. Give a corresponding physical interpretation of the result from (a). Further, what is the physical meaning of each of two conditions on b?

(10+10)