Engineering and Science Mathematics 2B

Homework 9

due May 8, 2013, before 12:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

1. Let A and B be two *statistically independent* events. Suppose $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$. Compute the probabilities $P(A|B)$, $P(B|A)$, $P(A \cup B)$, $P(A \cap B)$, $P(A - B)$, and $P(B - A)$.

2. A boy is selected at random from among the children belonging to families with n children.

   (E) What is the probability that the boy has $k - 1$ brothers?

   (A) It is known that the boy has at least two sisters. Show that the probability that he has $k - 1$ brothers is
   $$\frac{(n-1)!}{(2^{n-1} - n)\,(k-1)!\,(n-k)!}$$
   when $1 \leq k \leq n - 2$, and zero for other values of k.

   Hint: Use part (E) and Bayes' rule.

3. Gamblers A and B each have two unbiased four-sided dice, the four faces being numbered 1, 2, 3, 4. Without looking, B tries to guess the sum x of the numbers on the bottom faces of A's two dice after they have been thrown onto a table. If the guess is correct, B receives x^2 Euros, but if not he loses x Euros.

   (E) Show that, when guessing the sum of x, B's expected gain G per throw of A's dice is
   $$E[G_x] = p_x\,(x^2 + x) - x\,,$$
   where p_x is the probability that the sum of the bottom faces is x.

   (A) Compute the expected gain of B if he always guesses the sum of A's bottom faces from the previous round.

4. In how many ways can 8 people be placed around a table if there are three who insist on sitting together?

5. Prove the following identities:

   (E) $^{n}\mathrm{C}_{k}\ {}^{k}\mathrm{C}_{\ell} = {}^{n}\mathrm{C}_{\ell}\ {}^{n-l}\mathrm{C}_{k-\ell}$

   (A) $\sum_{i=0}^{k} {}^{m}\mathrm{C}_{i}\ {}^{n}\mathrm{C}_{k-i} = {}^{m+n}\mathrm{C}_{k}$

6. A royal family has children until it has a boy or until it has three children, whichever comes first. Assume that each child is a boy with probability $\frac{1}{2}$. Find the expected number of boys in this family and the expected number of girls.