Engineering and Science Mathematics 2B

Homework 7

due April 10, 2013, before 12:00

Normal questions and advanced questions (A) are worth 5 points; easy questions (E) are worth 4 points. Complete either the easy, or the advanced version, not both.

1. Let $f(x) = \sin(4x)$ on the interval $[0, 2\pi)$, periodically extended outside of this interval. Compute the complex Fourier coefficients c_k of f.

2. Find the Fourier transform of $f(x) = \mathrm{e}^{-|x|}$.

3. Show that the Fourier transform of $f(x + a)$ equals $\mathrm{e}^{\mathrm{i}a\xi} \tilde{f}(\xi)$.

4. (E) Show that $\widetilde{f'}(\xi) = \mathrm{i}\xi \, \tilde{f}(\xi)$.
   Hint: Integration by parts. You may assume that all boundary terms are zero when integrating by parts.

   (A) By taking the Fourier transform of the equation

$$\frac{d^2u}{dx^2} - u = f \,,$$

   show that the solution $u(x)$ can be written as

$$u(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\xi x} \, \tilde{f}(\xi)}{1 + \xi^2} \, \mathrm{d}\xi \,,$$

   where $\tilde{f}(\xi)$ is the Fourier transform of $f(x)$.

5. Compute the integral

   (E) $\displaystyle\int_{-2}^{2} \delta(2x) \, \cos x \, \mathrm{d}x \,,$

   (A) $\displaystyle\int_{-2\pi}^{2\pi} \delta(x^2 - \pi^2) \, \cos x \, \mathrm{d}x \,.$

6. (E) Let the *Heavyside* or *unit step function* be defined by

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0\,. \end{cases}$$

(The value at $x = 0$ is of no concern to us.)
Show that, for $x \neq 0$,

$$H(x) = \int_{-\infty}^{x} \delta(x)\,\mathrm{d}x\,.$$

(A) One can define (so-called distributional) derivatives of the δ-function via

$$\int_{-\infty}^{\infty} f(x)\,\delta^{(n)}(x)\,\mathrm{d}x = (-1)^n\, f^{(n)}(0)$$

for any n times differentiable function $f(x)$.
So we may be tempted to write a Taylor series for the δ-function,

$$\delta(x+a) = \sum_{n=0}^{\infty} \frac{\delta^{(n)}(x)}{n!}\, a^n\,.$$

It appears that the left side is zero except at $x = -a$, while the right side is zero except at $x = 0$. Resolve this paradox.