1. Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

(c) State a general result which guarantees that the computation in (b) can be successfully completed.

(10+5+5)

(a)

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 4-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(4-\lambda) + 2(-2)(-1) + (-1)(2)(-2)$$

$$- (-1)^2(4-\lambda) - (-2)^2(1-\lambda) - 2^2(1-\lambda)$$

$$= (1-2\lambda+\lambda^2)(4-\lambda) + 8 - 4 + \lambda - 8 + 8\lambda$$

$$= 4 - 8\lambda + 4\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 4 + 9\lambda$$

$$= 6\lambda^2 - \lambda^3$$

$$= \lambda^2(6-\lambda)$$

So the eigenvalues are $\lambda_1 = 6$ $\lambda_{2,3} = 0$ ("double eigenvalue")

Eigenvector for λ_1:

$$\begin{pmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{pmatrix} \xrightarrow[-5R3+R1 \to R3]{-R3 \to R1,\ 2R3+R2 \to R2} \begin{pmatrix} 1 & 2 & 5 \\ 0 & -6 & -12 \\ 0 & 12 & 24 \end{pmatrix} \xrightarrow[\frac{R2}{-6} \to R2,\ 2R2+R3 \to R3]{\frac{R2}{3}+R1 \to R1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Eigenvectors for $\lambda_{2,3}$: We simply need to find a basis for Ker A!

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix} \xrightarrow[2R3+R2 \to R2]{R1+R3 \to R3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $V_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $V_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are a basis for Ker A.

(b) $V_2 \perp V_1$ and $V_3 \perp V_1$, but $V_2 \not\perp V_3$.

Hence, we need to use Gram-Schmidt on $\{V_2, V_3\}$

Set $e_2 = \frac{V_2}{\|V_2\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$$\tilde{V}_3 = V_3 - \langle V_3, e_2 \rangle e_2$$
$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{5} \cdot 2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$e_3 = \frac{\tilde{V}_3}{\|\tilde{V}_3\|} = \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

Finally, setting $e_1 = \frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\{e_1, e_2, e_3\}$ is an ONB consisting of eigenvectors of A.

(c) A is symmetric $\Rightarrow$ A diagonalizable and the eigenvectors corresponding to distinct eigenvalues are orthogonal. For eigenspaces of higher dimension, we can always use Gram-Schmidt to get an ONB.

2. (E) Consider P_2, the vector space of polynomials of degree less or equal to two, endowed with inner product

$$\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx\,.$$

Show that the polynomials $p_1(x) = x-1$ and $p_2(x) = 4x^2+x-1$ are orthogonal. (8)

(A) Let V be an n-dimensional complex vector space with inner product $\langle\cdot,\cdot\rangle$ and orthonormal basis $e_1,\ldots,e_n$. Let $u, v \in V$ have coordinate vectors $a, b \in \mathbb{C}^n$ with respect to this basis. Show that

$$\langle u, v\rangle = a^H M b$$

where the matrix M is Hermitian and its matrix elements are given by

$$m_{ij} = \langle e_i, e_j\rangle\,.$$

(10)

(E) $$\langle p_1, p_2\rangle = \int_0^1 (x-1)(4x^2+x-1)\,dx$$

$$= \int_0^1 (4x^3 + x^2 - x - 4x^2 - x + 1)\,dx$$

$$= \frac{4}{4} + \frac{1}{3} - \frac{1}{2} - \frac{4}{3} - \frac{1}{2} + 1 = 0$$

(A) $$u = \sum_{i=1}^n a_i e_i \qquad v = \sum_{j=1}^n b_j e_j$$

$$\Rightarrow \langle u, v\rangle = \left\langle \sum_{i=1}^n a_i e_i\,, \sum_{j=1}^n b_j e_j\right\rangle = \sum_{i,j=1}^n a_i^* \underbrace{\langle e_i, e_j\rangle}_{=: m_{ij}} b_j$$

$$= a^H M b$$

3. Show that the Fourier transform of an odd function is odd.

(Recall that a function f is odd if $f(-x) = -f(x)$.) (10)

$$\tilde{f}(-\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-\xi)x} f(x)\, dx$$

$y = -x$

$\Rightarrow dy = -dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-i\xi y} \underbrace{f(-y)}_{= -f(y)} (-dy)$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi y} f(y)\, dy$$

$$= -\tilde{f}(\xi)$$

4. Let $f(x) = \sin x \cos x$.

(E) Compute the Fourier series of f on the interval $[0, 2\pi]$. (8)

(A) Compute the Fourier transform of f on $\mathbb{R}$. (10)

(E) $f(x) = \frac{1}{2} \sin 2x = \frac{e^{2ix} - e^{-2ix}}{4i}$

$\Rightarrow \quad c_2 = \frac{1}{4i} \quad , \quad c_{-2} = -\frac{1}{4i}$

$c_k = 0 \quad$ for $k \neq -2, 2$

(A) $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx$

$= \frac{1}{4i} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \left(e^{2ix} - e^{-2ix}\right) dx$

$= \frac{\sqrt{2\pi}}{4i} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix(2-\xi)}\, dx - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix(-2-\xi)}\, dx \right]$

$= \frac{\sqrt{2\pi}}{4i} \left(\delta(2-\xi) - \delta(-2-\xi) \right)$

$= \frac{\sqrt{2\pi}}{4i} \left(\delta(2-\xi) - \delta(2+\xi) \right)$

5. (E) Show that if a matrix S is real symmetric, then its eigenvalues are real. (8)

(A) Let V be a complex vector space with inner product $\langle \cdot, \cdot \rangle$. We say that a linear map $L\colon V \to V$ is *skew-Hermitian* provided

$$\langle Lv, w \rangle = -\langle v, Lw \rangle$$

for all $v, w \in V$.

Show that if λ is an eigenvalue of a skew-Hermitian map, then $\operatorname{Re}\lambda = 0$. (10)

(E) Suppose that λ is an eigenvalue with eigenvector v:

$$Sv = \lambda v$$

$$\Rightarrow \lambda^* \|v\|^2 = \lambda^* v^H v = (\lambda v)^H v = (Sv)^H v$$

$$= v^H S^H v = v^H (Sv) = v^H (\lambda v)$$

$$= \lambda \|v\|^2$$

$\Rightarrow \lambda^* = \lambda$, so λ is real.

(A) $$\lambda^* \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Lv, v \rangle = -\langle v, Lv \rangle$$

$$= -\langle v, \lambda v \rangle = -\lambda \langle v, v \rangle$$

$\Rightarrow \lambda^* = -\lambda$, so $\operatorname{Re}\lambda = 0$.

6. (E) Let c_k denote the complex Fourier coefficients of a 2π- periodic function f. Show that the Fourier coefficients of f' are given by $ik\,c_k$. (8)

(A) Consider the derivative operator $Lf = f'$ as a linear map on the vector space of smooth periodic functions on the interval $[0, 2\pi]$ endowed with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(x)\, g(x)\, dx\,.$$

Show that L is skew-Hermitian in the sense of Question 5A. Relate this fact to what you already know about the eigenvalues of the derivative operator. (10)

(E) $$f(x) = \sum_{k=-\infty}^{\infty} c_k\, e^{ikx}$$

$$f'(x) = \sum_{k=-\infty}^{\infty} \underbrace{ik\, c_k}_{\text{these are the Fourier coefficients of } f' \text{ by definition}}\, e^{ikx}$$

(A) $$\langle f', g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f'(x)^*\, g(x)\, dx$$

$$= \underbrace{\frac{1}{2\pi} f(x)^*\, g(x) \Big|_0^{2\pi}}_{=0 \text{ as } f,g \text{ are } 2\pi\text{-periodic}} - \frac{1}{2\pi} \int_0^{2\pi} f(x)^*\, g'(x)\, dx$$

$$= - \langle f, g' \rangle$$

This fits the definition of skew-Hermiticity from 5(A).

Since $(e^{ikx})' = ik\, e^{ikx}$, the numbers ik for k integer are the eigenvalues of $\frac{d}{dx}$; they are purely imaginary as concluded in 5(A

7. Compute the Fourier transform of $f(x) = e^{-|x|}$. (10)

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} e^{-|x|} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x(1-i\xi)} \, dx + \int_{0}^{\infty} e^{x(-1-i\xi)} \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-i\xi} e^{x(1-i\xi)} \Big|_{-\infty}^{0} + \frac{1}{-1-i\xi} e^{x(-1-i\xi)} \Big|_{0}^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-i\xi} + \frac{1}{1+i\xi} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2}$$