1. Let $A = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) For subspaces U and V of $\mathbb{R}^n$ we write $U = V^\perp$ if and only if $\operatorname{span}\{U, V\} = \mathbb{R}^n$ and $u^T v = 0$ for all $u \in U$ and $v \in V$.
Show that $\operatorname{Ker} A = (\operatorname{Range} A)^\perp$.

(c) Do you expect the result of part (b) to be true for any matrix $A \in M(3 \times 3)$? Explain why or why not.

(10+5+5)

(a) $$\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{3} - \lambda & \frac{2}{3} & 0 \\ \frac{2}{3} & -\lambda & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} - \lambda \end{vmatrix}$$

$$= \left(-\tfrac{1}{3} - \lambda\right)(-\lambda)\left(\tfrac{1}{3} - \lambda\right) + 2 \cdot \left(\tfrac{2}{3}\right)^2 \cdot 0 - \left(\tfrac{2}{3}\right)^2\left(-\tfrac{1}{3} - \lambda\right) - \left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3} - \lambda\right)$$

$$= -\lambda^3 + \tfrac{1}{9}\lambda \qquad + \tfrac{4}{27} + \tfrac{4}{9}\lambda - \tfrac{4}{27} + \tfrac{4}{9}\lambda$$

$$= \lambda\left(-\lambda^2 + 1\right)$$

$$= \lambda(1 - \lambda)(1 + \lambda)$$

So the eigenvalues are $\lambda = -1, 0, 1$

For $\lambda = -1$:

$$A - \lambda I = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{4}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow V_1 = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$$

For $\lambda = 0$:

$$A - \lambda I = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

For $\lambda = 1$:

$$A - \lambda I = \begin{pmatrix} -\frac{4}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & -1 & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow V_3 = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$$

(b) $u \in \text{Range } A \Rightarrow u = Aw$ for some $w \in V$. $v \in \text{Ker } A \Rightarrow Av = 0$

$\Rightarrow u^T v = (Aw)^T v = w^T A^T v = w^T A v = w^T 0 = 0$ $\square$

Here, we can also verify this statement concretely: V_1, V_2, V_3 are orthogonal.

Moreover, $\{V_1, V_3\}$ are a basis for Range A (see first midterm!)

$\{V_2\}$ is a basis for Ker A

Since $\{V_1, V_2, V_3\}$ is a basis for $\mathbb{R}^3$, $\text{Ker } A = (\text{Range } A)^\perp$

(or $\text{Range } A = (\text{Ker } A)^\perp$.)

(c) Eigenvectors of non-normal matrices are typically not orthogonal, so the argument breaks down. E.g.

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{Ker } A = \text{Range } A \neq (\text{Range } A)^\perp$ (first midterm!)

2. Find an orthonormal basis for P_2, the vector space of real polynomials of degree less or equal to two, endowed with inner product

$$\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx\,.$$

(10)

Use Gram-Schmidt on the basis $\{1, x, x^2\}$:

$b_1 = 1 \Rightarrow \|b_1\|^2 = \int_0^1 1\cdot 1\, dx = 1$

$\Rightarrow e_1 = b_1 = 1$ is already normalized.

Set $b_2 = x - \langle x, e_1\rangle e_1$ where $\langle x, e_1\rangle = \int_0^1 x\, dx = \frac{1}{2}$

$= x - \frac{1}{2}$

$\Rightarrow \|b_2\|^2 = \int_0^1 (x-\frac{1}{2})^2\, dx = \int_0^1 (x^2 - x + \frac{1}{4})\, dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$

$\Rightarrow e_2 = \sqrt{12}\,(x - \frac{1}{2})$

Set $b_3 = x^2 - \langle x^2, e_1\rangle e_1 - \langle x^2, e_2\rangle e_2$

where $\langle x^2, e_1\rangle = \int_0^1 x^2\, dx = \frac{1}{3}$ and $\langle x^2, x-\frac{1}{2}\rangle = \int_0^1 (x^3 - \frac{1}{2}x^2)\, dx = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

$\Rightarrow b_3 = x^2 - \frac{1}{3} - 12\,\frac{1}{12}\,(x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$

$\Rightarrow \|b_3\|^2 = \int_0^1 (x^4 + x^2 + \frac{1}{36} - 2x^3 - \frac{1}{3}x + \frac{1}{3}x^2)\, dx$

$= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{1}{2} - \frac{1}{6} + \frac{1}{9}$

$= \frac{1}{180}$

$\Rightarrow e_3 = \sqrt{180}\,(x^2 - x + \frac{1}{6})$

3. Let f be an odd function, i.e., $f(-x) = -f(x)$. Show that $\tilde{f}(0) = 0$. (10)

$$\tilde{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i0x} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{0} f(x)\, dx}_{= \int_0^\infty f(-x)\,dx \;=\; -\int_0^\infty f(x)\,dx} + \int_{0}^{\infty} f(x)\, dx \right] = 0 .$$

4. Let $f(x) = \sin^2 x \cos^2 x$.

(E) Compute the Fourier series of f on the interval $[0, 2\pi]$. (8)

(A) Compute the Fourier transform of f on $\mathbb{R}$. (10)

Note that $f(x) = \frac{1}{4}(1 - \cos 2x)(1 + \cos 2x)$

$$= \frac{1}{4}(1 - \cos^2 2x)$$

$$= \frac{1}{4}\left(1 - \frac{1}{2}(1 + \cos 4x)\right)$$

$$= \frac{1}{8}(1 - \cos 4x)$$

$$= \frac{1}{8} - \frac{1}{16}(e^{i4x} + e^{-i4x})$$

(E) We read off directly that $f_0 = \frac{1}{8}$, $f_4 = f_{-4} = -\frac{1}{16}$, and $f_k = 0$ for $k \neq 0, \pm 4$.

(A) We compute $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x}\left(\frac{1}{8} - \frac{1}{16}(e^{i4x} + e^{-i4x})\right) dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{e^{-i\xi x}}{8} - \frac{e^{i(4-\xi)x}}{16} - \frac{e^{i(-4-\xi)x}}{16}\right) dx$$

$$= \sqrt{2\pi}\left(\frac{\delta(\xi)}{8} - \frac{\delta(4-\xi)}{16} - \frac{\delta(4+\xi)}{16}\right)$$

5. (E) Show that if a matrix S is real symmetric, then eigenvectors corresponding to distinct eigenvalues are orthogonal. (8)

(A) Let V be a complex vector space with inner product $\langle\cdot,\cdot\rangle$. We say that a linear map $L\colon V \to V$ is *skew-Hermitian* provided

$$\langle Lv, w\rangle = -\langle v, Lw\rangle$$

for all $v, w \in V$.

Show that the eigenvectors corresponding to distinct eigenvalues of a skew-Hermitian map are orthogonal. (10)

(E) Suppose $Sv_i = \lambda_i v_i$ for $i=1,2$ with $\lambda_1 \neq \lambda_2$.

Then $v_1^T S v_2 = v_1^T \lambda_2 v_2 = \lambda_2 v_1^T v_2$ (1)

On the other hand, since S is symmetric,

$$v_1^T S v_2 = (Sv_1)^T v_2 = \lambda_1 v_1^T v_2 \qquad (2)$$

Comparing (1) and (2), we conclude that $v_1^T v_2 = 0$.

(A) This is really the same argument as for (E):

Suppose $Lv_i = \lambda_i v_i$ for $i = 1,2$ with $\lambda_1 \neq \lambda_2$.

Then $<Lv_1, v_2> = <\lambda_1 v_1, v_2> = \lambda_1^* <v_1, v_2>$

$$\begin{array}{l} \| \\ -<v_1, Lv_2> = -<v_1, \lambda_2 v_2> = -\lambda_2 <v_1, v_2> \end{array}$$

Moreover, the eigenvalues of a skew-Hermitian map are purely imaginary (proof left as exercise), so the above implies

$$(\lambda_1 - \lambda_2)<v_1, v_2> = 0 \qquad \Rightarrow \quad <v_1, v_2> = 0.$$

Note: The cases (E) and (A) are both special cases of a normal map!

6. Let f and g be 2π-periodic complex-valued functions whose complex Fourier coefficients are denoted f_k and g_k, respectively. Write

$$(f \otimes g)(x) = \int_0^{2\pi} f(y)^* \, g(x+y) \, dy$$

to denote their cross-correlation function. Show that the Fourier coefficients of $f \otimes g$ are given by $2\pi f_k^* g_k$.

Write $f(x) = \sum_{k=-\infty}^{\infty} f_k e^{ikx}$ and $g(x) = \sum_{j=-\infty}^{\infty} g_j e^{ijx}$

$$\Rightarrow \quad (f \otimes g)(x) = \int_0^{2\pi} \sum_{k=-\infty}^{\infty} f_k^* e^{-iky} \sum_{j=-\infty}^{\infty} g_j e^{ij(x+y)} \, dy$$

$$= \sum_{k,j=-\infty}^{\infty} f_k^* g_j e^{ijx} \underbrace{\int_0^{2\pi} e^{i(j-k)y} \, dy}_{= 2\pi \delta_{jk}}$$

$$= \sum_{k=-\infty}^{\infty} 2\pi f_k^* g_k e^{ikx}$$

This expression identifies $2\pi f_k^* g_k$ as the Fourier coefficients of $f \otimes g$.

7. Compute the Fourier transform of $f(x) = 1 + H(x)\,e^{-x}$, where H denotes the Heavyside function. (10)

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \left(1 + H(x) e^{-x}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x-i\xi x}\, dx$$

$$= \sqrt{2\pi}\, \delta(\xi) + \frac{1}{\sqrt{2\pi}} \frac{1}{-1-i\xi} e^{-x-i\xi x} \Big|_{0}^{\infty}$$

$$= \sqrt{2\pi}\, \delta(\xi) + \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\xi}$$