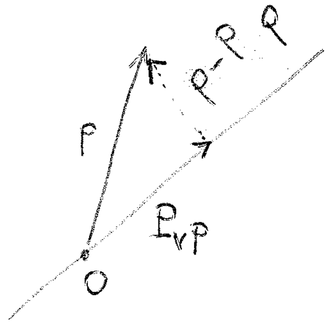
1. Let $\mathbf{v} = (-1, 1, 1)^{\mathsf{T}}$.

(E) Find the distance of the point $\mathbf{p} = (0, 1, 0)^{\mathsf{T}}$ to the line through the origin in the direction of $\mathbf{v}$.

(8)

(A) Find the matrix representation in the standard basis of the projection onto the plane through the origin which is normal to $\mathbf{v}$. (10)



(E) $$d = |P_v p - p|$$

$$= |\hat{v}\,\hat{v}\cdot p - p|$$

$$= \left|\frac{1}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix}\begin{pmatrix}-1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}\right|$$

$$= \left|\left(-\tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{1}{3}\right)\right| = \sqrt{\tfrac{1}{9} + \tfrac{4}{9} + \tfrac{1}{9}} = \frac{\sqrt{6}}{3}$$

(A) <u>Solution 1:</u>

$Qp = p - P_v p$ from (E) is just the result of this projection, so its matrix representation follows directly:

$$Qp = p - P_v p = Ip - \hat{v}\hat{v}^{\mathsf{T}} p = (I - \hat{v}\hat{v}^{\mathsf{T}})p$$

$$= \left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix}\begin{pmatrix}-1&1&1\end{pmatrix}\right] p$$

$$= \begin{pmatrix}\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\end{pmatrix} p$$

Solution 2:

Use the cross product to find an ON set of basis vectors spanning the plane:

$$u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

is clearly perpendicular to v, so

$$w = u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 - 0 \\ 0 - 1\cdot 1 \\ 1 - (-1) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

is perpendicular to both. So the projection reads

$$Qp = \hat{u}\hat{u}^T p + \hat{w}\hat{w}^T p = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} p + \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & -4 \end{pmatrix} p$$

$$= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} p$$

Solution 3: A basis for the plane is a basis for the kernel of P_v, i.e. the solution space for the linear system corresponding to

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{so} \quad u = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \text{ and } w = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

However, u and w are not orthogonal, so we better orthonormalize them using the Gram-Schmidt procedure.

$$\hat{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

$$x = w - P_{\hat{u}} w = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} (-1 \; -1 \; 0) \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\hat{x} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

Now $\hat{u}$ and $\hat{x}$ form an ON basis for span $\{u, w\}$, and

$$Qp = \hat{u}\hat{u}^T p + \hat{x}\hat{x}^T p .$$

The rest of the computation proceeds as for Solution 2.

2. Find the general solution to the system of linear equations $Ax = b$ with

(E)

$$A = \begin{pmatrix} 4 & 4 & 1 & 7 \\ 3 & 3 & 0 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix}.$$

(16)

(A)

$$A = \begin{pmatrix} 3-i & 0 & 1+3i \\ 2-i & -1 & -1+5i \\ i & i & 2+2i \end{pmatrix}, \qquad b = \begin{pmatrix} 10 \\ 7 \\ 3i \end{pmatrix}.$$

(20)

Check your answer! (Required for full credit.)

(E)

$$\left(\begin{array}{cccc|c} 4 & 4 & 1 & 7 & -5 \\ 3 & 3 & 0 & 6 & -6 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow[\substack{R2-3R3 \to R2 \\ R1-4R3 \to R3}]{R3 \to R1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -3 & 3 & -9 \\ 0 & 0 & -3 & 3 & -9 \end{array}\right)$$

$$\xrightarrow[-\frac{1}{3}R2 \to R2]{R2-R3 \to R3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R1-R2 \to R1} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad x = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

Check:

$$A \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -8+3 \\ -6+0 \\ -2+3 \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix} \quad \checkmark$$

$$A \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \ \checkmark \qquad A \begin{pmatrix} 2 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 8-1-7 \\ 6+0-6 \\ 2-1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \ \checkmark$$

(A)

$$\left(\begin{array}{ccc|c} 3-i & 0 & 1+3i & 10 \\ 2-i & -1 & -1+5i & 7 \\ i & i & 2+2i & 3i \end{array}\right) \xrightarrow[R3+R2\to R3]{\substack{-iR3\to R1 \\ R3+R1\to R2}} \left(\begin{array}{ccc|c} 1 & 1 & 2-2i & 3 \\ 3 & i & 3+5i & 10+3i \\ 2 & -1+i & 1+7i & 7+3i \end{array}\right)$$

$$\xrightarrow[-2R1+R3\to R3]{-3R1+R2\to R2} \left(\begin{array}{ccc|c} 1 & 1 & 2-2i & 3 \\ 0 & -3+i & -3+11i & 1+3i \\ 0 & -3+i & -3+11i & 1+3i \end{array}\right) \xrightarrow[R2-R3\to R3]{(3+i)R2\to R2} \left(\begin{array}{ccc|c} 1 & 1 & 2-2i & 3 \\ 0 & -10 & -20+30i & 10i \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{R2}{10}+R1\to R1]{\frac{R2}{-10}\to R2} \left(\begin{array}{ccc|c} 1 & 0 & i & 3+i \\ 0 & 1 & 2-3i & -i \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad x = \begin{pmatrix} 3+i \\ -i \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} i \\ 2-3i \\ -1 \end{pmatrix}$$

Check:

$$A\begin{pmatrix} 3+i \\ -i \\ 0 \end{pmatrix} = \begin{pmatrix} (3+i)(3-i)+0+0 \\ (3+i)(2-i)+i+0 \\ (3+i)i - i^2 \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \\ 3i \end{pmatrix} \checkmark$$

$$A\begin{pmatrix} i \\ 2-3i \\ -1 \end{pmatrix} = \begin{pmatrix} i(3-i) - (1+3i) \\ i(2-i) - (2-3i) + 1-5i \\ +i^2 + i(2-3i) - (2+2i) \end{pmatrix} = \begin{pmatrix} 3i+1-1-3i \\ 2i+1-2+3i+1-5i \\ -1+2i+3-2-2i \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

3. (E) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Show that $\operatorname{Ker} A = \operatorname{Range} A$. (8)

(A) Let $A, B \in M(n \times n)$ with $\operatorname{Ker} A \cap \operatorname{Range} B = \{0\}$. Show that $\dim \operatorname{Ker} AB = \dim \operatorname{Ker} B$. (10)

(E) Range A is the space spanned by the column vectors of A,

so here Range $A = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

Ker A is the solution space of the linear system $Ax = 0$.

Here: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ Ker $A = \operatorname{span}\left\{\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right\}$

We see that Ker A = Range A.

(A) We show that Ker AB = Ker B.

Let $v \in \operatorname{Ker} B \Rightarrow Bv = 0 \Rightarrow ABv = 0 \Rightarrow v \in \operatorname{Ker} AB$.

Let $v \notin \operatorname{Ker} B \Rightarrow Bv \neq 0$

as $Bv \in \operatorname{Range} B$ and $\operatorname{Ker} A \cap \operatorname{Range} B = \{0\}$, we conclude $Bv \notin \operatorname{Ker} A$

$\Rightarrow ABv \neq 0$

$\Rightarrow v \notin \operatorname{Ker} AB$

This altogether proves that Ker B = Ker AB

4. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Write out a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$.

(c) Check your result by explicitly performing the matrix multiplications SD and AS.

(10+5+5)

(a) $p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

has zeros $\pm i$.

For $\lambda_1 = i$: $A - \lambda_1 I = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \rightarrow \begin{pmatrix} 1 & +i \\ 0 & 0 \end{pmatrix}$

$\Rightarrow v_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}$

For $\lambda_2 = -i$: $A - \lambda_2 I = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$

$\Rightarrow v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

(b) $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\qquad S = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$

(c) $SD = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -i & -i \end{pmatrix}$

$AS = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -i & -i \end{pmatrix}$ ✓

5. (E) Let $A \in M(n \times n)$ be a regular matrix and suppose that λ is an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}. (8)

(A) Let $A \in M(n \times n)$ be diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct) and $v_1, \ldots, v_n$ corresponding linearly independent eigenvectors. How can you choose, given this information, a basis for Range A? Explain why your method works. (10)

(E) Since A is invertible, $\lambda \neq 0$.

Let v be the corresponding eigenvector, so

$$Av = \lambda v \Rightarrow v = A^{-1}(\lambda v) \Rightarrow \frac{1}{\lambda} v = A^{-1} v$$

$\Rightarrow \frac{1}{\lambda}$ is an eigenvalue of A^{-1} with eigenvector v

(A) $\text{Range } A = \text{span}\{v_i : \lambda_i \neq 0\}$

Note that $A = SDS^{-1}$ with $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $S = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

$\Rightarrow \text{Range } A = \{SDS^{-1}x : x \in \mathbb{R}^n\}$

$= \{SDy : y \in \mathbb{R}^n\}$ since S invertible

$= \left\{ \begin{pmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{pmatrix} y : y \in \mathbb{R}^n \right\}$

$= \text{span}\{\lambda_1 v_1, \ldots, \lambda_n v_n\}$

$= \text{span}\{v_i : \lambda_i \neq 0\}$

Since the $\{v_i\}$ are l.i., they are a basis of their span.

6. Let V be the set of symmetric 2×2 real matrices.

(a) Show that V is a vector space with the usual matrix addition and scalar multiplication.

(b) Find a basis B for the vector space V. (Keep it simple, do not use a basis containing the matrices I, E, S from below!)

(c) Show that $B' = \{I, E, S\}$ where I is the identity matrix,

$$E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

is another basis V. Compute the change of coordinate matrices $I_{B',B}$ and $I_{B,B'}$.

(d) Show that for any fixed $C \in M(2 \times 2)$ the map F defined by $F(A) = CAC^T$ is a linear map on V.

(e) Give the matrix which represents F with respect to a basis of your choice.

(5+5+10+5+5)

(a) Since the usual matrix addition and scalar multiplication satisfy the correct algebraic relations in the vector space of 2x2 matrices, it suffices to show that they map into the symmetric matrices:

$$\left.\begin{aligned} (A+B)^T &= A^T + B^T = A+B \\ (\lambda A)^T &= \lambda A^T = A \end{aligned}\right\} \text{ for } A,B \in V$$

(b) A symmetric 2x2 matrix A has the single constraint $a_{12} = a_{21}$,

so $$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = a_{11} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=: e_1} + a_{12} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=: e_2} + a_{22} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{=: e_3}$$

$$\Rightarrow B = \{e_1, e_2, e_3\}$$

(c) Write the coordinates w.r.t. $\mathcal{B}$ of I, E, S into the columns of a matrix:

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Let's compute the inverse:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

We see that M is invertible $\Rightarrow$ the coordinate vectors of I, E, S are l.i. $\Rightarrow$ I, E, S themselves are l.i.

$\Rightarrow$ $\mathcal{B}' = \{I, E, S\}$ is a basis as it's the correct number of l.i. vectors.

We further conclude that $I_{\mathcal{B}',\mathcal{B}} = M$ and $I_{\mathcal{B},\mathcal{B}'} = M^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

(d) Linearity is clear from the rules of matrix arithmetic.

(Let's check anyway: $F(A+B) = C(A+B)C^T$

$$= CAC^T + CBC^T = F(A) + F(B)$$

$$F(\lambda A) = C(\lambda A)C^T = \lambda CAC^T = \lambda F(A).$$)

What needs to be checked, though, is that $\mathcal{F}$ maps into V, i.e., that $\mathcal{F}(A)$ is symmetric when A is symmetric:

$$\mathcal{F}(A)^T = (CAC^T)^T = (C^T)^T A^T C^T = CAC^T = \mathcal{F}(A).$$

(e) Let's coordinatise $\mathcal{F}$ with respect to the basis B: We compute

$$\mathcal{F}(e_1) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix}}_{= \begin{pmatrix} c_{11} & c_{21} \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} c_{11}^2 & c_{11}c_{21} \\ c_{11}c_{21} & c_{21}^2 \end{pmatrix}$$

$$\mathcal{F}(e_2) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix}}_{= \begin{pmatrix} c_{12} & c_{22} \\ c_{11} & c_{21} \end{pmatrix}} = \begin{pmatrix} 2c_{11}c_{12} & c_{21}c_{12}+c_{11}c_{22} \\ c_{21}c_{12}+c_{11}c_{22} & 2c_{22}c_{21} \end{pmatrix}$$

$$\mathcal{F}(e_3) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix}}_{\begin{pmatrix} 0 & 0 \\ c_{12} & c_{22} \end{pmatrix}} = \begin{pmatrix} c_{12}^2 & c_{12}c_{22} \\ c_{12}c_{22} & c_{22}^2 \end{pmatrix}$$

$$\Rightarrow \quad M_{\mathcal{F}} = \begin{pmatrix} c_{11}^2 & 2c_{11}c_{12} & c_{12}^2 \\ c_{11}c_{21} & c_{21}c_{12}+c_{11}c_{22} & c_{12}c_{22} \\ c_{21}^2 & 2c_{22}c_{21} & c_{22}^2 \end{pmatrix}$$