Engineering and Science Mathematics 2B

Midterm II Retake

April 22, 2013

For questions with an easy (E) and an advanced (A) version, choose either the easy or the advanced version, not both. You may make a different choice for each question.

Useful identities:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cos^2\theta = \tfrac{1}{2}\,(1+\cos 2\theta) \qquad \sin^2\theta = \tfrac{1}{2}\,(1-\cos 2\theta)$$

$$\cos\theta\,\sin\theta = \tfrac{1}{2}\,\sin 2\theta$$

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x}\, f(x)\, dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\, \tilde{f}(\xi)\, d\xi$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\, d\xi$$

$$c_k = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{e}^{-2\pi \mathrm{i} k x/L}\, f(x)\, \mathrm{d}x \qquad f(x) = \sum_{k=-\infty}^{\infty} c_k\, \mathrm{e}^{2\pi \mathrm{i} k x/L}$$

1. Let $A = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

   (a) Find the eigenvalues and eigenvectors of A.

   (b) For subspaces U and V of $\mathbb{R}^n$ we write $U = V^\perp$ if and only if $\operatorname{span}\{U, V\} = \mathbb{R}^n$ and $u^T v = 0$ for all $u \in U$ and $v \in V$.
   Show that $\operatorname{Ker} A = (\operatorname{Range} A)^\perp$.

   (c) Do you expect the result of part (b) to be true for any matrix $A \in M(3 \times 3)$? Explain why or why not.

(10+5+5)

2. Find an orthonormal basis for P_2, the vector space of real polynomials of degree less or equal to two, endowed with inner product

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, \mathrm{d}x\,.$$

(10)

3. Let f be an odd function, i.e., $f(-x) = -f(x)$. Show that $\tilde{f}(0) = 0$. (10)

4. Let $f(x) = \sin^2 x \cos^2 x$.

   (E) Compute the Fourier series of f on the interval $[0, 2\pi]$. (8)

   (A) Compute the Fourier transform of f on $\mathbb{R}$. (10)

5. (E) Show that if a matrix S is real symmetric, then eigenvectors corresponding to distinct eigenvalues are orthogonal. (8)

   (A) Let V be a complex vector space with inner product $\langle \cdot, \cdot \rangle$. We say that a linear map $L \colon V \to V$ is *skew-Hermitian* provided

$$\langle Lv, w \rangle = -\langle v, Lw \rangle$$

   for all $v, w \in V$.

   Show that the eigenvectors corresponding to distinct eigenvalues of a skew-Hermitian map are orthogonal. (10)

6. Let f and g be 2π-periodic complex-valued functions whose complex Fourier coefficients are denoted f_k and g_k, respectively. Write

$$(f \otimes g)(x) = \int_0^{2\pi} f(y)^*\, g(x+y)\, \mathrm{d}y$$

   to denote their cross-correlation function. Show that the Fourier coefficients of $f \otimes g$ are given by $2\pi\, f_k^*\, g_k$. (10)

7. Compute the Fourier transform of $f(x) = 1 + H(x)\, \mathrm{e}^{-x}$, where H denotes the Heavyside function. (10)