Engineering and Science Mathematics 2B

Midterm II

April 17, 2013

For questions with an easy (E) and an advanced (A) version, choose either the easy or the advanced version, not both. You may make a different choice for each question.
Useful identities:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta) \qquad \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$\cos\theta\,\sin\theta = \tfrac{1}{2}\sin 2\theta$$

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x}\, f(x)\, dx \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\, \tilde{f}(\xi)\, d\xi$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\, d\xi$$

$$c_k = \frac{1}{L} \int_{-L/2}^{L/2} \mathrm{e}^{-2\pi \mathrm{i} k x/L}\, f(x)\, \mathrm{d}x \qquad f(x) = \sum_{k=-\infty}^{\infty} c_k\, \mathrm{e}^{2\pi \mathrm{i} k x/L}$$

1. Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$.

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

   (c) State a general result which guarantees that the computation in (b) can be successfully completed.

(10+5+5)

2. (E) Consider P_2, the vector space of polynomials of degree less or equal to two, endowed with inner product

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, \mathrm{d}x\,.$$

Show that the polynomials $p_1(x) = x - 1$ and $p_2(x) = 4\,x^2 + x - 1$ are orthogonal. (8)

(A) Let V be an n-dimensional complex vector space with inner product $\langle \cdot, \cdot \rangle$ and orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. Let $\boldsymbol{u}, \boldsymbol{v} \in V$ have coordinate vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^n$ with respect to this basis. Show that

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{a}^H M \boldsymbol{b}$$

where the matrix M is Hermitian and its matrix elements are given by

$$m_{ij} = \langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle \,.$$

(10)

3. Show that the Fourier transform of an odd function is odd.

   (Recall that a function f is odd if $f(-x) = -f(x)$.) (10)

4. Let $f(x) = \sin x \, \cos x$.

   (E) Compute the Fourier series of f on the interval $[0, 2\pi]$. (8)

   (A) Compute the Fourier transform of f on $\mathbb{R}$. (10)

5. (E) Show that if a matrix S is real symmetric, then its eigenvalues are real. (8)

   (A) Let V be a complex vector space with inner product $\langle \cdot, \cdot \rangle$. We say that a linear map $L \colon V \to V$ is *skew-Hermitian* provided

$$\langle Lv, w \rangle = -\langle v, Lw \rangle$$

   for all $v, w \in V$.

   Show that if λ is an eigenvalue of a skew-Hermitian map, then $\operatorname{Re} \lambda = 0$. (10)

6. (E) Let c_k denote the complex Fourier coefficients of a 2π-periodic function f. Show that the Fourier coefficients of f' are given by $\mathrm{i}k \, c_k$. (8)

   (A) Consider the derivative operator $Lf = f'$ as a linear map on the vector space of smooth periodic functions on the interval $[0, 2\pi]$ endowed with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*(x) \, g(x) \, \mathrm{d}\boldsymbol{x} \,.$$

   Show that L is skew-Hermitian in the sense of Question 5A. Relate this fact to what you already know about the eigenvalues of the derivative operator. (10)

7. Compute the Fourier transform of $f(x) = \mathrm{e}^{-|x|}$. (10)