Engineering and Science Mathematics 2B

Midterm I

March 6, 2013

For questions with an easy (E) and an advanced (A) version, choose either the easy or the advanced version, not both. You may make a different choice for each question.

1. (E) Let $A \in M(n \times n)$ be a regular matrix and suppose that λ is an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}. (8)

   (A) Let $A \in M(n \times n)$ be diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct) and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ corresponding linearly independent eigenvectors. How can you choose, given this information, a basis for Range A? Explain why your method works. (10)

2. Let $A = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$.

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Write out a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$.

   (c) Check your result by explicitly performing the matrix multiplications SD and AS.

   (10+5+5)

3. Let $\boldsymbol{v} = (1, 1, -1)^T$.

   (E) Find the distance of the point $\boldsymbol{p} = (0, 1, 0)^T$ to the line through the origin in the direction of $\boldsymbol{v}$.

   (A) Find the matrix representation in the standard basis of the projection onto the plane through the origin which is normal to $\boldsymbol{v}$. (10)

4. Find the general solution to the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ with

   (E)

$$A = \begin{pmatrix} 1 & 7 & 4 & 4 \\ 0 & 6 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} -5 \\ -6 \\ 1 \end{pmatrix}.$$

   (16)

(A)

$$A = \begin{pmatrix} 0 & 1+3i & 3-i \\ -1 & -1+5i & 2-i \\ i & 2+2i & i \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 10 \\ 7 \\ 3i \end{pmatrix}.$$

(20)

Check your answer! (Required for full credit.)

5. Let V be the set of symmetric 2×2 real matrices.

   (a) Show that V is a vector space with the usual matrix addition and scalar multiplication.

   (b) Find a basis B for the vector space V. (Keep it simple, do not use a basis containing the matrices I, E, S from below!)

   (c) Show that $B' = \{I, E, S\}$ where I is the identity matrix,

   $$E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

   is another basis V. Compute the change of coordinate matrices $I_{B',B}$ and $I_{B,B'}$.

   (d) Show that for any fixed $C \in M(2 \times 2)$ the map F defined by $F(A) = CAC^T$ is a linear map on V.

   (e) Let

   $$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   Give the matrix which represents F with respect to a basis of your choice.

(5+5+10+5+5)

6. (E) Consider the matrix

   $$A = \begin{pmatrix} 0 & 0 \\ 7 & 0 \end{pmatrix}.$$

   Show that $\operatorname{Ker} A = \operatorname{Range} A$. (8)

   (A) Let $A, B \in M(n \times n)$ with $\operatorname{Ker} A \cap \operatorname{Range} B = \{0\}$. Show that $\dim \operatorname{Ker} AB = \dim \operatorname{Ker} B$. (10)