1. Show that the complete graph with 5 vertices cannot be embedded in the plane. (10)

The valency of each vertex is 4, so

$$4\,|V| = 2\,|E| \quad \Rightarrow \quad |E| = 10 .$$

So, by Euler's Formula, if the graph were embeddable,

$$|F| = 2 + |E| - |V| = 2 + 10 - 5 = 7 .$$

On the other hand, each edge bounds two faces and each face is bounded by at least three edges, so

$$3\,|F| \leq 2\,|E| \quad \Rightarrow \quad |F| \leq \frac{2 \cdot 10}{3} < 7 .$$

Contradiction, so the complete graph with 5 vertices cannot be embedded in the plane.

2. Show that a finite graph is bipartite if and only if it does not contain a cycle of odd length.

(Recall that a graph is bipartite if its vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that every edge has one vertex in V_1 and the other vertex in V_2.) (5+5)

"$\Rightarrow$": Suppose the graph is bi-partite and contains a cycle. Pick a starting vertex on the cycle, say in V_1, and traverse the cycle. Then after an even number of steps, you have a vertex in V_1 and after an odd number of steps, you have a vertex in V_2 as with each step you must cross into the opposite set. Since you must come out in V_1 in the end, the number of edges in the cycle must be even.

"$\Leftarrow$": WLOG suppose the graph is connected. (Otherwise, apply the argument to each connected component separately.)

Pick a vertex v_1. Let d denote the usual distance function, and set

$$V_1 = \{v \in V : d(v_1, v) \text{ is even}\}$$

$$V_2 = \{v \in V : d(v_1, v) \text{ is odd}\}$$

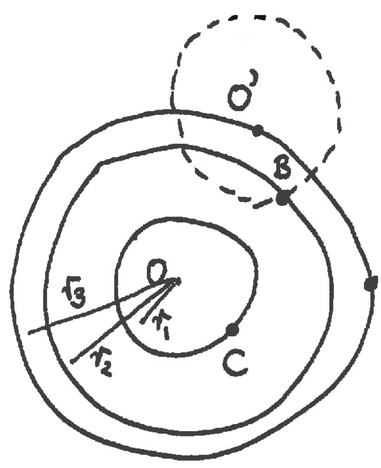
Suppose these two sets do not bi-partition the graph, i.e., there is an edge between some vertices u and w in the same set. But then u and w are connected to v by paths of both even or both odd lengths, so that we have a cycle of odd length altogether! Contradiction.

3. (a) Given three concentric circles, how can you construct an equilateral triangle with one vertex on each of the circles?

(b) Give a necessary and sufficient condition for the existence of such an equilateral triangle.

(5+5)

(a)



- Pick any point on outer circle, for example. Call it A.
- Rotate inner circle by 60° (consequently, the rotated center O' lies on outer circle).
- Denote the point of intersection with middle circle B, and the pre-image of B under rotation by C
- Then $|\overline{AB}| = |\overline{AC}|$ by construction, and $\measuredangle CAB = 60°$
- Hence ΔABC is equilateral.

(b) This works whenever the point of intersection B exists, i.e.

$$r_3 - r_1 \leq r_2 .$$

Vice versa, if a solution to the problem exists, you can perform the indicated construction, so condition is necessary too.

4. Consider a transformation of the plane written in vector form as

$$F(v) = Mv \qquad \text{where} \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(a) Show that F is a motion of the plane (i.e., preserves lengths) if and only if

$$a^2 + c^2 = 1\,,$$
$$b^2 + d^2 = 1\,,$$
$$ab + cd = 0\,.$$

(b) What is the geometric significance of these conditions when you consider the vectors

$$u = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \text{and} \qquad w = \begin{pmatrix} b \\ c \end{pmatrix}\ ?$$

(5+5)

(a) Write $v = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\|F(v)\| = \|v\| \iff \|Mv\|^2 = \|v\|^2 \iff v^T M^T M v - v^T v = 0$$

$$\iff 0 = (ax+by \quad cx+dy)\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} - (x^2+y^2)$$

$$= a^2x^2 + 2abxy + b^2y^2 + c^2x^2 + cdxy + d^2y^2 - x^2 - y^2$$

$$= x^2(a^2+b^2-1) + 2xy(ab+cd) + y^2(b^2+d^2-1)$$

This holds true for all $x, y \in \mathbb{R}$ if and only if the coefficients of x^2, xy, and y^2 are all zero.

(b) u, v are unit vectors and are perpendicular.

(I.e., $\{u, v\}$ is an ONB of $\mathbb{R}^2$.)

5. When G is a group, the cyclic group generated by some $a \in G$ is called a *group cycle*. Now construct a graph whose vertices are the the elements of G. Insert an edge whenever two of the group elements are adjacent in one of the group cycles (ordered naturally). This graph is known as the *cycle graph*.[1]

   (a) Prove that the cycle graph of a finite group is connected.

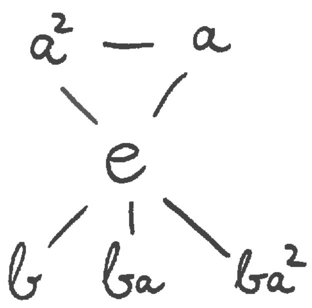
   (b) Draw the cycle graph for the dihedral group

$$D_3 = \{\langle a, b \rangle : a^3 = e, b^2 = e, ab = ba^{-1}\}.$$

(5+5)

(a) Every group cycle contains e, so e is connected to every group element.

(b)



[1]The usual definition restricts to *primitive cycles*, those which do not appear as a subset of another cycle, but this shall not matter here.

6. Suppose the symmetry group G of an ornament contains H_0, the point reflection about the origin, and the glide reflection U, chosen such that U^2 is the translation by one unit along the x-axis.

Show that G must contain point reflections about all points $(n/2, 0)$ with $n \in \mathbb{Z}$ and line reflections about all lines $x = 1/4 + n/2$ with $n \in \mathbb{Z}$. (10)

$$(x,y) \xmapsto{H_0} (-x,-y) \xmapsto{U} \left(-x+\tfrac{1}{2}, y\right)$$

$$(x,y) \xmapsto{T_{-\frac{1}{4}}} \left(x-\tfrac{1}{4}, y\right) \xmapsto{R_0} \left(\tfrac{1}{4}-x, y\right) \xmapsto{T_{\frac{1}{4}}} \left(\tfrac{1}{2}-x, y\right)$$

So we see that $U \circ H_0$ is the reflection about the line $x=\frac{1}{4}$. Since H_0 is a symmetry, there must be a corresponding axis of symmetry at $x=-\frac{1}{4}$, and by translational symmetry therefore about all lines $x = \frac{1}{4} + \frac{n}{2}$, $n \in \mathbb{Z}$.

Further, since there is a reflection symmetry about the line $x=\frac{1}{4}$, the point symmetry H_0 is mapped to a point symmetry about $(\frac{1}{2}, 0)$ and by translation to all points $(\frac{n}{2}, 0)$, $n \in \mathbb{Z}$.

Alternatively, you can verify by direct computation that

$$H_{n/2} = U^{2n} H_0 \quad \text{and} \quad R_{\frac{1}{4}+\frac{n}{2}} = U^{2n+1} H_0 .$$