# General Mathematics and Computational Science II

Midterm Exam

March 14, 2012

1. Show that the complete graph with 5 vertices cannot be embedded in the plane. (10)

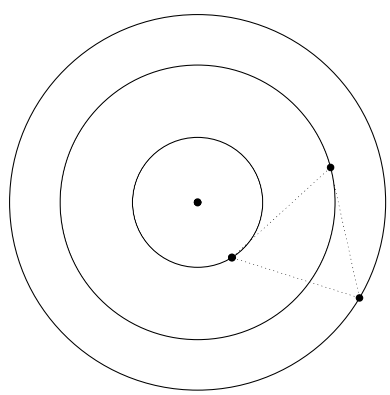
2. Show that a finite graph is bipartite if and only if it does not contain a cycle of odd length.

   (Recall that a graph is bipartite if its vertex set can be partitioned into two disjoint subsets $V_1$ and $V_2$ such that every edge has one vertex in $V_1$ and the other vertex in $V_2$.) (5+5)

3. (a) Given three concentric circles, how can you construct an equilateral triangle with one vertex on each of the circles?

   (b) Give a necessary and sufficient condition for the existence of such an equilateral triangle.

   (5+5)



4. Consider a transformation of the plane written in vector form as

$$F(\boldsymbol{v}) = \mathsf{M}\boldsymbol{v} \qquad \text{where} \qquad \mathsf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

(a) Show that $F$ is a motion of the plane (i.e., preserves lengths) if and only if

$$\begin{aligned} a^2 + c^2 &= 1\,, \\ b^2 + d^2 &= 1\,, \\ ab + cd &= 0\,. \end{aligned}$$

(b) What is the geometric significance of these conditions when you consider the vectors

$$\boldsymbol{u} = \begin{pmatrix} a \\ c \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{w} = \begin{pmatrix} b \\ d \end{pmatrix}\,?$$

(5+5)

5. When $G$ is a group, the cyclic group generated by some $a \in G$ is called a *group cycle*. Now construct a graph whose vertices are the elements of $G$. Insert an edge whenever two of the group elements are adjacent in one of the group cycles (ordered naturally). This graph is known as the *cycle graph*.[1]

   (a) Prove that the cycle graph of a finite group is connected.

   (b) Draw the cycle graph for the dihedral group

$$D_3 = \{\langle a, b\rangle \colon a^3 = e, b^2 = e, ab = ba^{-1}\}\,.$$

(5+5)

6. Suppose the symmetry group $G$ of an ornament contains $H_0$, the point reflection about the origin, and the glide reflection $U$, chosen such that $U^2$ is the translation by one unit along the $x$-axis.

   Show that $G$ must contain point reflections about all points $(n/2, 0)$ with $n \in \mathbb{Z}$ and line reflections about all lines $x = 1/4 + n/2$ with $n \in \mathbb{Z}$. (10)

[1] The usual definition restricts to *primitive cycles*, those which do not appear as a subset of another cycle, but this shall not matter here.