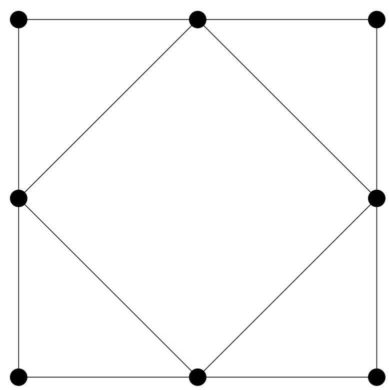
General Mathematics and Computational Science II

Final Exam

May 25, 2012

1. Can you draw a path in the plane which crosses each edge of the following graph exactly once and crosses none of the vertices (marked as dots)? (10)



2. Let ℓ denote the line through the origin in the direction of unit vector $\boldsymbol{u}$.

   Show that the reflection of a point with coordinates $\boldsymbol{p}$ about ℓ is given by

$$R_\ell(\boldsymbol{p}) = 2\,\boldsymbol{u}\,\boldsymbol{u}^T\boldsymbol{p} - \boldsymbol{p}\,.$$

(10)

3. (a) Show that symmetric group S_3, the group of permutations of a three-element set, and the dihedral group D_3, the group of symmetries of an equilateral triangle, are isomorphic.

   (b) Are S_4 and D_4 isomorphic?

   (The symmetric group S_4 denotes the group of permutations of a four-element set and the dihedral group D_4 denotes the group of symmetries of a square.)

(10+5)

4. Consider the linear programming problem of minimizing

$$z = -3\,x - 2\,y$$

subject to

$$
\begin{aligned}
x + 3\,y &\leq 15\,,\\
4\,x + y &\leq 16\,,\\
x, y &\geq 0\,.
\end{aligned}
$$

(a) Solve this problem *using the simplex method.*

(b) Write out the dual problem and solve the dual problem *using the graphical method.*

(c) State the relation you expect between (a) and (b) and verify that your solution meets your expectation.

(10+10+5)

5. Assume that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ are two solutions to the linear programming problem in standard form,

$$
\begin{aligned}
&\text{minimize } \boldsymbol{c}^T \boldsymbol{x}\\
&\text{subject to } A\boldsymbol{x} = \boldsymbol{b}\\
&\text{and } \boldsymbol{x} \geq \boldsymbol{0}\,.
\end{aligned}
$$

Show that any convex combination $\boldsymbol{z} = t\,\boldsymbol{x} + (1-t)\,\boldsymbol{y}$ for $t \in [0,1]$ is also a solution.

(10)

6. Recall the definition of the discrete Fourier transform of the N-tuple of complex numbers $v_0, \ldots, v_{N-1}$,

$$\tilde{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}jkh}\, v_j$$

with $h = 2\pi/N$ for $k = 0, \ldots, N-1$.

Compute the discrete Fourier transform when N is even and

$$v_j = \begin{cases} 1 & \text{for } j < N/2\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

Hint: The case $k = 0$ is special, but if you think of your expression for $\hat{v}_k$ as being defined for arbitrary $k \in \mathbb{R}$, you should find that $\lim_{k\to 0} \hat{v}_k = \hat{v}_0$. (There is no need to compute this, but you may want to use it as a consistency check.) (10)

7. For a nonnegative integer a of at most N decimal digits, let $a_0, \ldots, a_{N-1}$ denote its decimal digits, so that

$$a = \sum_{i=0}^{N-1} a_i\, 10^i\,,$$

likewise for nonnegative integers b and c.

(a) Suppose that $c = a \cdot b$, write out an expression for c_i in terms of a_i and b_i, where $i = 0, \ldots, N-1$.

(b) Given your answer to (a), how can you use the Fast Fourier Transform to speed up the multiplication of large integers?

(c) How should N be chosen such that your proposed algorithm from part (b) computes a product of integers correctly?

(10+5+5)