General Mathematics and CPS II

Exercise 20

May 4, 2012

1. Solve the difference equation

$$t_n = 2\,t_{n-1} + 2^{n+1}\,,$$
$$t_1 = 0\,.$$

2. Let $u\colon [0, 2\pi] \to \mathbb{R}$ be defined via the Fourier series

$$u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k\, \mathrm{e}^{\mathrm{i}kx}\,.$$

   Show that its derivative $w(x) = u'(x)$, assuming it exists, has Fourier coefficients

$$\hat{w}_k = \mathrm{i}k\,\hat{u}_k\,. \qquad (*)$$

3. The previous question suggests a way to differentiate a function $u\colon [0, 2\pi] \to \mathbb{R}$ numerically: First, sample u on the equidistant grid $x_j = jh$ with $h = 2\pi/N$. Second, compute the Fourier coefficients of its trigonometric interpolant v as described in Section 3 of the handout. Third, use (*) on the Fourier coefficients $\hat{v}_k$. Finally, take the inverse DFT to obtain the derivative sampled on the grid.

   A practical difficulty lies in the convention that standard FFT software libraries take the indexing convention $j = 0, \ldots, N-1$ and $k = 0, \ldots, N-1$. In particular, the input and output data arrays are arranged in precisely this order. Trigonometric interpolation, on the other hand, requires that $k = -N/2, \ldots, N/2 - 1$. With which factor, then, must the ith element, with respect to the order imposed by the software, of the vector $\hat{\boldsymbol{v}}$ be multiplied so that differentiation is computed correctly?