1. (a) Solve the partial differential equation

$$u_t + t\,u_x = 0\,,$$

where $u = u(x,t)$ using the method of characteristics.

(b) Draw the characteristic curves, then state a set of boundary and/or initial conditions that specify the solution uniquely in the first quadrant of the (x,t) plane.

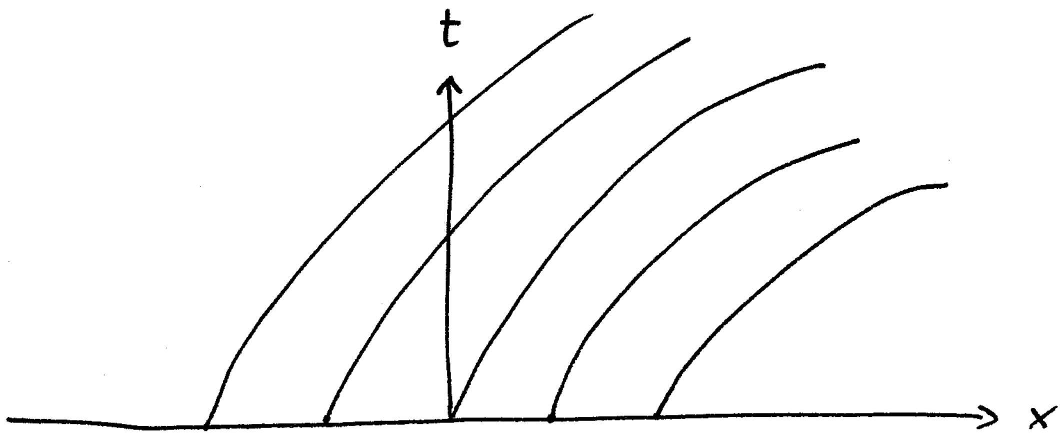
(5+5)

(a) $$\frac{d}{dt}\,u(x(t),t) = u_t(x(t),t) + \dot{x}(t)\,u_x(x(t),t)$$

Thus, $\frac{d}{dt}\,u(x(t),t) = 0$ if $\dot{x}(t) = t$

i.e., if $x(t) = x_0 + \frac{1}{2}t^2$

(b)



$\Rightarrow$ The characteristics are parabolas in t, so on $\mathbb{R}$, need to specify initial condition $u(x,0) = g(x)$ so that $u(x,t) = g(x - \frac{1}{2}t^2)$.

For the solution in the first quadrant, need data on positive x and t axes.

2. (a) Show that

$$\Delta u = 2n \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \fint_{\partial B(x,\varepsilon)} (u(y) - u(x))\, dS(y).$$

(b) Let

$$U^+ = \{x \in \mathbb{R}^n : 0 < x_1 < 1, |x_2| < 1, \ldots, |x_n| < 1$$

with $n \geq 2$ denote an open half-cube. Suppose $u \in C(\bar{U}^+)$ is harmonic in U^+ with $u(0, x_2, \ldots, x_n) = 0$ for $|x_2| \leq 1, \ldots, |x_n| \leq 1$.

Show, by referring to the result from part (a) or otherwise, that

$$v(x) = \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(-x_1, x_2, \ldots, x_n) & \text{for } x_1 < 0 \end{cases}$$

is harmonic in the open cube $|x_1| < 1, \ldots, |x_n| < 1$.

(5+5)

(a) Let $\phi(\varepsilon) = \fint_{\partial B(x,\varepsilon)} u(y)\, dS(y) \longrightarrow u(x)$ as $\varepsilon \to 0$

Note: $\phi(\varepsilon) = \frac{1}{\mathrm{Vol}(\partial B(x,\varepsilon))} \int_{\partial B(x,1)} u(\varepsilon z)\, \varepsilon^{n-1}\, dS(z)$

$$= \frac{1}{\mathrm{Vol}(\partial B(x,1))} \int_{\partial B(x,1)} u(\varepsilon z)\, dS(z)$$

$$\Rightarrow \phi'(\varepsilon) = \frac{1}{\mathrm{Vol}(\partial B(x,1))} \int_{\partial B(x,1)} z \cdot Du(\varepsilon z)\, dS(z)$$

$$= \fint_{\partial B(x,\varepsilon)} \overset{= \nu}{\frac{y}{\varepsilon}} \cdot Du(y)\, dS(y)$$

$$= \underbrace{\frac{\mathrm{Vol}(B(x,\varepsilon))}{\mathrm{Vol}(\partial B(x,\varepsilon))}}_{= \varepsilon/n} \fint_{B(x,\varepsilon)} \Delta u\, dy$$

Now use L'Hôpital's rule:

$$\lim_{\varepsilon\to 0} \frac{\phi(\varepsilon)-\phi(0)}{\varepsilon^2} = \lim_{\varepsilon\to 0} \frac{\phi'(\varepsilon)}{2\varepsilon} = \frac{1}{2n} \lim_{\varepsilon\to 0} \fint_{B(x,\varepsilon)} \Delta u \, dy = \frac{1}{2n}\Delta u$$

This proves the claim.

(b) $\Delta v = 0$ for $x_1 > 0$ by assumption

$\Delta v = 0$ for $x_1 < 0$ by direct verification.

So we need to show that $\Delta v = 0$ when $x_1 = 0$.

So let $x = (0, x_2, \ldots, x_n)$. Then

$$\int_{\partial B(x,\varepsilon)} v \, dS(y) = \int_{\partial B(x,\varepsilon)\cap U^+} u(y) \, dS(y) + \int_{\partial B(x,\varepsilon)\setminus U^+} (-u(-y_1, y_2 \ldots)) \, dS(y)$$

$$= 0$$

Since $v(x) = 0$ by assumption, part (a) implies that $\Delta v = 0$ in this case as well.

3. Recall that the solution to the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)\,,$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t=0\}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy\,,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|z|^2}{4t}}\,.$$

Suppose that $g \in L^1(\mathbb{R}^n)$.

(a) Show that $||u||_{L^\infty} \to 0$ as $t \to \infty$.

(b) Show that, for all $t \geq 0$,

$$\int_{\mathbb{R}^n} u(x,t)\, dx = \text{const}\,.$$

(c) Give a physical interpretation of (a) vs. (b).

(3+3+4)

(a) $$|u(x,t)| \leq \underbrace{\int_{\mathbb{R}^n} |g(y)|\, dy}_{= \|g\|_{L^1} < \infty} \cdot \underbrace{\sup_{z \in \mathbb{R}^n} \phi(0,t)}_{= \frac{1}{(4\pi t)^{n/2}} \to 0 \text{ as } t \to \infty}$$

(b) $$\int_{\mathbb{R}^n} u(x,t)\, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy\, dx$$
$$= \int_{\mathbb{R}^n} g(y) \underbrace{\int_{\mathbb{R}^n} \Phi(x-y,t)\, dx}_{=1}\, dy = \int_{\mathbb{R}^n} g(y)\, dy$$

(c) If u is a concentration, then $\int_{\mathbb{R}^n} u(x,t)\,dx$ is the total mass.
Thus, (b) is saying that a diffusion process preserves total mass.

4. Consider the wave equation

$$
\begin{aligned}
u_{tt} - u_{xx} = 0 \qquad & \text{in } \mathbb{R} \times (0, \infty), \\
u = g \qquad & \text{on } \mathbb{R} \times \{t = 0\}, \\
u_t = h \qquad & \text{on } \mathbb{R} \times \{t = 0\}
\end{aligned}
$$

for $g \in C^2$ and $h \in C^1$. Derive d'Alembert's solution formula

$$
u(x, t) = \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, dy .
$$

Note: A constructive derivation is required for full credit.

Hint: Factorize the wave equation as $(\partial_t + \partial_x)\underbrace{(\partial_t - \partial_x)u}_{\equiv v} = 0$.

(10)

$V_t + V_x = 0 \quad \Rightarrow \quad v(x,t) = v(x-t,0)$

$\Rightarrow \quad u_t - u_x = v(x-t,0)$

$\Rightarrow \quad \frac{d}{ds} u(x-s, t+s) = v(x-2s-t, 0) = h(x-2s-t) - g'(x-2s-t)$

$$
\begin{aligned}
\Rightarrow \quad u(x,t) - \underbrace{u(x+t,0)}_{=g(x+t)} &= \int_{-t}^{0} v(x-2s-t,0)\, ds \\
&= -\frac{1}{2} \int_{x+t}^{x-t} v(y)\, dy \\
&= \frac{1}{2} \int_{x-t}^{x+t} h(y)\, dy - \frac{1}{2} \underbrace{\int_{x-t}^{x+t} g'(y)\, dy}_{= g(x+t) - g(x-t)}
\end{aligned}
$$

$\Rightarrow \quad u(x,t) = \frac{1}{2}\left(g(x+t) - g(x-t) + \int_{x-t}^{x+t} h(y)\, dy\right)$

5. (a) Let $u \in C_1^3(\mathbb{R} \times [0,\infty))$ solve the *Airy equation*

$$\begin{aligned} u_t + u_{xxx} &= 0 \quad \text{in } \mathbb{R} \times (0,\infty)\,, \\ u(x,0) &= g(x) \quad \text{on } \mathbb{R} \times \{t=0\}\,, \end{aligned}$$

and suppose that $u, u_x \to 0$ as $x \to \pm\infty$ and u_{xx} is bounded.
Prove that u is the unique solution in this class.
Hint: Energy methods.

(b) Extend your uniqueness proof to the *Korteweg–de Vries equation*

$$\begin{aligned} u_t + u u_x + u_{xxx} &= 0 \quad \text{in } \mathbb{R} \times (0,\infty)\,, \\ u(x,0) &= g(x) \quad \text{on } \mathbb{R} \times \{t=0\}\,. \end{aligned}$$

(5+5)

We prove (b) only; the proof of (a) is a strict simplification thereof.

Let $\hat{u}$ and $\bar{u}$ solve the KdV equation; set $w = \hat{u} - \bar{u}$.

Then w satisfies the equation

$$w_t + w\hat{u}_x + \bar{u} w_x + w_{xxx} = 0$$

Now set $E = \frac{1}{2}\int_{\mathbb{R}} w^2\,dx$ and compute

$$\dot{E} = \int_{\mathbb{R}} w w_t\,dx = -\int_{\mathbb{R}} w^2 \hat{u}_x\,dx - \underbrace{\int_{\mathbb{R}} w\bar{u}w_x} - \int_{\mathbb{R}} w w_{xxx}\,dx$$

$$= \frac{1}{2}\int_{\mathbb{R}} \bar{u}(w^2)_x = \underbrace{\frac{1}{2}\bar{u}w^2\Big|_{-\infty}^{\infty}}_{=0} - \frac{1}{2}\int_{\mathbb{R}} \bar{u}_x w^2\,dx$$

and $$\int_{\mathbb{R}} w w_{xxx}\,dx = \underbrace{w w_{xx}\Big|_{-\infty}^{\infty}}_{=0} - \frac{1}{2}\int_{\mathbb{R}} (w_x^2)_x\,dx = -\frac{1}{2}(w_x)^2\Big|_{-\infty}^{\infty} = 0$$

$$\Rightarrow \dot{E} \leq 2\,\|\hat{U}_x\|_{L^\infty} E + \|\hat{U}_x\|_{L^\infty} E \equiv cE$$

Now integrate from 0 to t, noting the monotonicity of the integral:

$$\int_0^t \frac{\dot{E}}{E}\,dt \leq \int_0^t c\,dt$$

$$\Rightarrow \ln \frac{E(t)}{E(0)} \leq ct$$

$$\Rightarrow \quad E(t) \leq E(0)\,e^{ct}$$

Since $E(0)=0$ and E is nonnegative, we conclude $E(t)=0 \quad \forall\, t \geq 0$, hence

$$\hat{U} = \bar{U}.$$

$\square$