Introduction to Partial Differential Equations

Midterm Exam

April 12, 2011

1. (a) Solve the partial differential equation

$$u_t + t\, u_x = 0\,,$$

where $u = u(x,t)$ using the method of characteristics.

(b) Draw the characteristic curves, then state a set of boundary and/or initial conditions that specify the solution uniquely in the first quadrant of the (x,t) plane.

(5+5)

2. (a) Show that

$$\Delta u = 2n \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} ⨍_{\partial B(x,\varepsilon)} (u(y) - u(x))\, dS(y)\,.$$

(b) Let

$$U^+ = \{x \in \mathbb{R}^n \colon 0 < x_1 < 1, |x_2| < 1, \dots, |x_n| < 1$$

with $n \geq 2$ denote an open half-cube. Suppose $u \in C(\bar{U}^+)$ is harmonic in U^+ with $u(0, x_2, \dots, x_n) = 0$ for $|x_2| \leq 1, \dots, |x_n| \leq 1$.

Show, by referring to the result from part (a) or otherwise, that

$$v(x) = \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(-x_1, x_2, \dots, x_n) & \text{for } x_1 < 0 \end{cases}$$

is harmonic in the open cube $|x_1| < 1, \dots, |x_n| < 1$.

(5+5)

3. Recall that the solution to the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)\,,$$
$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}$$

is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy\,,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|z|^2}{4t}}\,.$$

Suppose that $g \in L^1(\mathbb{R}^n)$.

(a) Show that $\|u\|_{L^\infty} \to 0$ as $t \to \infty$.

(b) Show that, for all $t \geq 0$,

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \text{const}\,.$$

(c) Give a physical interpretation of (a) vs. (b).

(3+3+4)

4. Consider the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad \text{in } \mathbb{R} \times (0,\infty)\,, \\ u &= g \qquad \text{on } \mathbb{R} \times \{t=0\}\,, \\ u_t &= h \qquad \text{on } \mathbb{R} \times \{t=0\} \end{aligned}$$

for $g \in C^2$ and $h \in C^1$. Derive d'Alembert's solution formula

$$u(x,t) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy\,.$$

Note: A constructive derivation is required for full credit.

Hint: Factorize the wave equation as $(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0$.

(10)

5. (a) Let $u \in C_1^3(\mathbb{R} \times [0,\infty))$ solve the *Airy equation*

$$\begin{aligned} u_t + u_{xxx} &= 0 \quad \text{in } \mathbb{R} \times (0,\infty)\,, \\ u(x,0) &= g(x) \quad \text{on } \mathbb{R} \times \{t=0\}\,, \end{aligned}$$

and suppose that $u, u_x \to 0$ as $x \to \pm\infty$.
Prove that u is the unique solution in this class.
Hint: Energy methods.

(b) Extend your uniqueness proof to the *Korteweg–de Vries equation*

$$\begin{aligned} u_t + u\,u_x + u_{xxx} &= 0 \quad \text{in } \mathbb{R} \times (0,\infty)\,, \\ u(x,0) &= g(x) \quad \text{on } \mathbb{R} \times \{t=0\}\,. \end{aligned}$$

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