Introduction to Partial Differential Equations

Final Exam

May 25, 2011

1. (a) Solve the partial differential equation

$$t\,u_t + x\,u_x = 0\,,$$
$$u(x,1) = g(x)\,,$$

where $u = u(x,t)$ using the method of characteristics for $t \geq 1$.

(b) Draw the characteristic curves. Discuss how you could solve the equation in the exterior of the unit disk.

(5+5)

2. Suppose u is a radial solution of the *Helmholtz equation*

$$u - \Delta u = 0 \qquad \text{in } \mathbb{R}^n\,.$$

Set $u(x) = v(r)$ with $r = |x|$. Show that v must satisfy the *modified Bessel equation*

$$v - (n-1)\,\frac{v'}{r} - v'' = 0\,.$$

(Do not attempt to solve it.) (10)

3. Let $U \subset \mathbb{R}^n$ be open, bounded, and connected. Let $u\colon U \to \mathbb{R}$ be a nonnegative harmonic function. Show that if $u(x) = 0$ for some $x \in U$, then $u = 0$ everywhere in U. (10)

4. Show that if the initial datum to the heat equation is even, the solution will be even for any fixed $t \geq 0$. (5)

5. Suppose $u = u(x,t)$ is a smooth function on $\mathbb{R} \times [0,\infty)$ with compact support in the x-direction for every fixed $t \geq 0$. Suppose further that u solves Burgers' equation

$$u_t + F(u)_x = 0 \qquad \text{with} \qquad F(u) = \tfrac{1}{2}\,u^2\,.$$

(a) Show that

$$M(t) = \int_{\mathbb{R}} u(x,t)\,dx \qquad \text{and} \qquad E(t) = \int_{\mathbb{R}} u(x,t)^2\,dx$$

are constants of the motion.

(b) Show that

$$\int_0^\infty \int_{\mathbb{R}} (u\,v_t + F(u)\,v_x)\,dx\,dt + \int_{\mathbb{R}} u(x,0)\,v(x,0)\,dx = 0$$

for every $v \in C^1(\mathbb{R} \times [0,\infty))$ with compact support.

(5+5)

6. *Note:* This problem continues Question 5 which should be attempted beforehand.

Suppose that u is an integral solution of Burgers' equation (i.e., it satisfies the condition stated in 5b) which is smooth everywhere except on a curve C. Suppose further that C has a smooth parametrization of the form $(s(t), t)$. Let u_l denote the left-hand limit of u on C and let u_r denote the right-hand limit of u on C.

(a) Show that, on C,

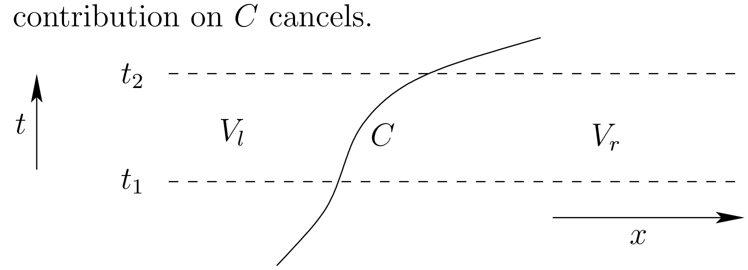
$$[[F(u)]] = \dot{s}\,[[u]]$$

where $[[F(u)]] = F(u_l) - F(u_r)$ and $[[u]] = u_l - u_r$ denote the jump in $F(u)$ and the jump in u across C, respectively.

Hint: Use the divergence theorem.

(b) Show that $M(t)$ is a constant of the motion.

Hint: Use the divergence theorem on V_l and V_r separately and notice that the contribution on C cancels.



(c) Suppose u satisfies the *entropy condition*

$$u(x+h,t) - u(x,t) \le \frac{c}{t}\,h$$

for some constant $c > 0$. Show that this implies $u_l \ge u_r$.

(d) Show that, when u satisfies the entropy condition, $E(t)$ is a decreasing function of time.

Hint: Show that in the interior of V_l and V_r,

$$\tfrac{1}{2}\,\partial_t u^2 + \tfrac{1}{3}\,\partial_x u^3 = 0\,.$$

Now use the divergence theorem as in (b) and discuss the sign of the contribution on C.

(e) Give a physical interpretation of (d) vs. (b).

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