General Mathematics and ACM II

Midterm Exam

March 10, 2011

1. In a finite graph, define the distance $d(u,v)$ between two vertices u and v as the length of the shortest path between u and v. Show that d satisfies the "triangle inequality"

$$d(u,v) + d(v,w) \geq d(u,w)$$

   for all vertices u, v, and w. (5)

2. (a) Show that every simple graph with at least two vertices has two vertices of same valency.

   (Recall that a simple graph is a graph without multiple edges between pairs of vertices.)

   (b) Give an example that this statement is false if you allow multiple edges.

   (5+5)

3. Let G be a finite connected planar graph with V its set of vertices, E its set of edges, and F its set of faces.

   (a) Show that $2\,|E| \geq 3\,|F|$.

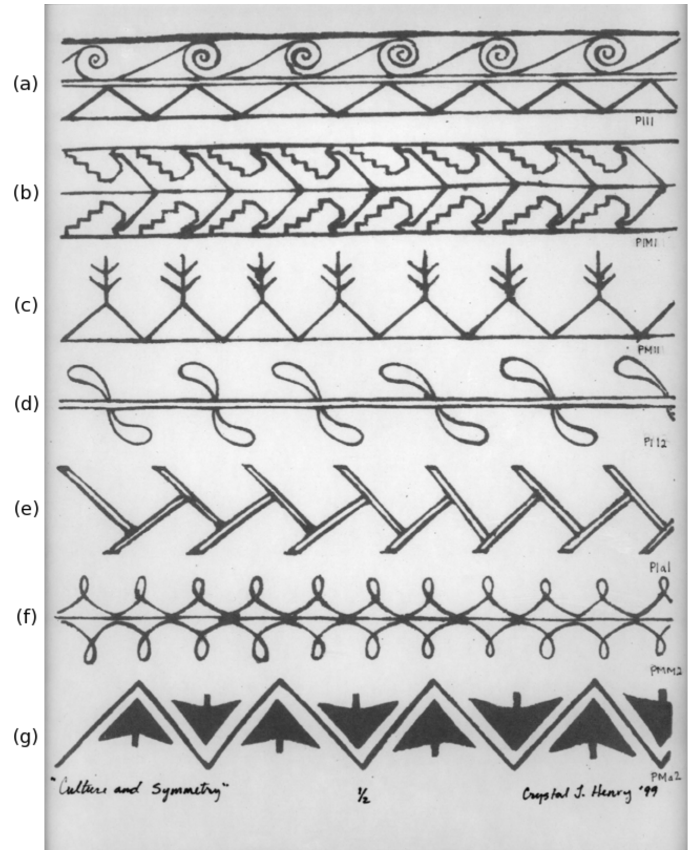
   (b) Show that $|E| \leq 3\,|V| - 6$.

   (c) Conclude that every planar graph with less than 12 vertices must have at least one vertex of valency less than 5.

   (5+5+5)

4. Show that a quadrilateral (a polygon with four edges) is a trapezoid (a quadrilateral with two parallel edges) if and only if the length of the line segment joining the midpoints of a pair of opposite edges is equal to half the sum of the lengths of the other two edges. (10)

5. Describe the symmetry group for each of the following ornaments.



Iroquois and Ojibwa border designs. From `http://www.oswego.edu/~baloglou/103/crystal.html`

(10)

6. Recall that the finite cyclic group of order n is

$$\begin{aligned} C_n &= \{\langle a \rangle \colon a^n = e\} \\ &= \{e, a, \ldots, a^{n-1}\} \end{aligned}$$

and that the symmetry group of an n-gon is the dihedral group

$$\begin{aligned} D_n &= \{\langle a, b \rangle \colon a^n = e, b^2 = e, ab = ba^{-1}\} \\ &= \{e, a, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1}\}\,. \end{aligned}$$

Show that any finite subgroup of the motions of the plane (length-preserving transformations of the plane) is isomorphic to either C_n or D_n for some $n \geq 1$.

Note: This is a typical classification problem where you have to sift through all possibilities. Do not expect a short answer. Full credit will be given if the delineation of the problem is reasonably complete. Extra credit for a complete solution. (10)