Stochastic Processes

Homework 2

Due in class Thursday, March 4

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_i\}_{i \in I}$ a partition of Ω into pairwise disjoint measurable sets. Prove *Bayes' Theorem*, namely that

$$P(A_i|B) = \frac{P(B|A_i)\, P(A_i)}{\sum_{j \in I} P(B|A_j)\, P(A_j)}$$

for every $i \in I$ and $B \in \mathcal{F}$.

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let ξ and η be independent random variables. Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be Borel functions. Show that $f(\xi)$ and $g(\eta)$ are independent.

   (Recall that f is a Borel function if $f^{-1}(B)$ is a Borel set for every Borel set B.)

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space, ξ a random variable, and η a discrete random variable taking values $\{y_i\}_{i \in I}$ where I is at most countable. Show that

$$E(\xi) = \sum_{i \in I} E(\xi | \eta^{-1}\{y_i\})\, P\eta^{-1}\{y_i\}\,.$$

4. Two dice are tossed. What is the conditional expectation $E(\xi|\eta)$ of the total amount ξ shown given η, the absolute value of the difference of the amounts shown?

5. Let $\Omega = [-1/2, 1/2]$ with the Lebesgue measure. Find the conditional expectation $E(\xi|\eta)$ if

$$\xi(x) = x\,, \qquad \eta(x) = x^2\,.$$