Stochastic Processes

Homework 1

Due in class Thursday, February 25

Each problem is worth 5 points.

1. Show that any open subset of $\mathbb{R}$ is a countable union of open intervals.

2. Show that

   (a) all one-point sets $\{x\}$ with $x \in \mathbb{R}$

   (b) $\mathbb{Q}$

   belong to $\mathcal{B}(\mathbb{R})$, the Borel σ-field on $\mathbb{R}$.

3. Let $\Omega = [0,1]$ and $A \subsetneq \Omega$ nonempty. What is the σ-field $\mathcal{F}$ generated by $\{A\}$? What are the $\mathcal{F}$-measurable functions?

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\xi \colon \Omega \to \mathbb{R}$ a random variable which is nonnegative a.e., in other words, $P\{x \in \Omega \colon \xi(x) < 0\} = 0$. Show that

$$\int_\Omega \xi \, \mathrm{d}P \geq 0 \,.$$

5. Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$, and P the Lebesgue measure. Give an example of a sequence of nonnegative Borel functions $\{f_n\}_{n \in \mathbb{N}}$ defined on $[0,1]$ such that

$$\int_\Omega f_n \, \mathrm{d}P \to 0 \,,$$

   but $f_n \nrightarrow 0$ on a set of positive measure.