Perspectives of Mathematics II

Homework 2

Due in class Monday, March 15

1. Listen to the excellent lecture by Gilbert Strang of MIT on the singular value decomposition, which is available at

   `http://ocw.mit.edu/OcwWeb/Mathematics/18-06Spring-2005/VideoLectures/detail/lecture29.htm`

   At the end of his first example (16–27 minutes into the lecture), Strang notices a sign problem which he cannot resolve on the spot. It is not due to an accidental miscalculation. Can you help Prof. Strang and explain the wrong sign?

2. Consider the Tychonov regularization of an ill-conditioned system of linear equations, see the class notes for details. Show that the Tychonov regularized solution $\boldsymbol{x}_\alpha$ minimizes the function
   $$f_\alpha(\boldsymbol{x}) = \|A\boldsymbol{x} - \boldsymbol{b}\|^2 + \alpha \|\boldsymbol{x}\|^2$$
   over all $\boldsymbol{x} \in \mathbb{R}^n$.

3. Replace the range condition used in class for the error analysis of the Tychonov regularization of an ill-conditioned system of linear equations by the stronger condition

   $$\boldsymbol{x}_{\text{true}} = A^T A \boldsymbol{z}$$

   for some $\boldsymbol{z}$ whose norm is comparable to the norm of $\boldsymbol{x}_{\text{true}}$. Redo the error analysis to find an expression for the optimal choice of α as a function of $\|\boldsymbol{\eta}\|$ and $\|\boldsymbol{z}\|$ under this stronger range condition.

4. Consider the matrix $A \in M(n \times n)$ with components

   $$a_{ij} = \frac{n - |i - j|}{n^2} .$$

   Use a computer program to compute its singular values $\sigma_1, \ldots, \sigma_n$ when n is large. Formulate a conjecture how the singular value σ_i depends on i over large ranges of the index.

   Hint: Cf. Homework 1, Problem 3.

5. The data set located at

was generated by applying the blur matrix A of the previous problem (with $n = 200$), then adding Gaussian random noise $\boldsymbol{\eta}$ with $\|\boldsymbol{\eta}\| = 0.001$. Obtain a reconstruction of the original data set using *a posteriori* selection of the regularization parameter. You should send in your program and the reconstructed data in plain ASCII format by email.