Perspectives of Mathematics II

Homework 1

Due in class Wednesday, February 24

Each problem is worth 5 points. Except for the first two problems, the tasks are computational. There is no need to write programs bottom-up; it is recommended to use any of the standard high level mathematical software packages.

1. Let $\mathcal{V} = \mathbb{R}^n$ and $A \in M(n \times n)$ be a symmetric matrix. Show that the maximum of

$$f(\boldsymbol{v}) = \boldsymbol{v}^T A \boldsymbol{v}$$

on the unit ball of $\mathcal{V}$ is attained when $\boldsymbol{v}$ is a principal eigenvector of A. (A principal eigenvector is an eigenvector corresponding to the largest eigenvalue.)

2. Let $\mathcal{V} = C^2([-1, 1])$. Find the minimum of

$$f(u) = \int_{-1}^{1} u'(x)^2 \, \mathrm{d}x$$

on $\mathcal{V}$ subject to the constraint $g(u) = 0$ with

$$g(u) = \int_{-1}^{1} u(x) \, \mathrm{d}x - 2 \, .$$

3. You know from Calculus that for a smooth function on the reals,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x + h/2) - f(x - h/2)}{h} \, .$$

Thus, we expect these fractions, for finite values of h, to approximate $f'(x)$. Test out the quality of these approximations by considering the errors

$$E_1(h) = \left| f'(0) - \frac{f(h) - f(0)}{h} \right|$$

and

$$E_2(h) = \left| f'(0) - \frac{f(h/2) - f(-h/2)}{h} \right|$$

for the function $f(x) = \exp(x)$ within a reasonable range of values for h. Fit the model

$$E(h) = c\,h^p$$

to the data obtained for each $E_1(h)$ and $E_2(h)$. What values for c and p do you obtain in each case? Which approximation is better?

4. The United States National Institute of Standards and Technology (NIST) publishes reference data sets for use in validating numerical software. Use the software of your choice to reproduce the linear least square fit to a polynomial of degree two of the *Pontius* data set, available at

   `http://www.itl.nist.gov/div898/strd/lls/data/Pontius.shtml`

   Compute the polynomial coefficients and compare with the published values. Plot the data and the graph of your fitted model.

5. Repeat the previous task with the *Filip* data set, available at

   `http://www.itl.nist.gov/div898/strd/lls/data/Filip.shtml`

   Why is this set considered difficult?

6. Perform a nonlinear least square fit on the *Misra1a* data set, available at

   `http://www.itl.nist.gov/div898/strd/nls/data/misra1a.shtml`

   Compute the coefficients of the exponential model described therein and compare with the published values. Plot the data and the graph of your fitted model.

7. *(Up to 15 points extra credit.)* Consider the model for the rise in sea level due to the melting of small ice caps and glaciers, equation (1) in the 2008 MCM paper by Evans and Stepien. Track the estimation of the model coefficients τ and β though the literature. Can you reproduce these estimates or validate with old or even newer data?