Functional Analysis

Homework 9

May 28, 2009

1. Let E, F be Banach spaces and let $A\colon \mathcal{D}(A) \subset E \to F$ be a linear, closed, densely defined, unbounded operator.

   Show that A is invertible if and only if

   (i) There exists $\alpha > 0$ such that for all $u \in \mathcal{D}(A)$

   $$\|Au\|_F \geq \alpha \, \|u\|_E \, .$$

   (ii) Range A is dense in F.

   Then, moreover, $\|A^{-1}\|_{\mathcal{L}(F,E)} \leq \dfrac{1}{\alpha}$.

2. Let $H = \ell^2(\mathbb{N})$.

   (a) Show that the operator defined via $Ae_n = n^{-1}\, e_n$ for $n \in \mathbb{N}$, where $\{e_n\}$ is the canonical basis in ℓ^2, is compact.

   (b) What is $\sigma(A)$?

   (c) Let R be the right shift operator on ℓ^2, i.e. $R(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$. What is $\sigma(RA)$?

3. (a) Let E, F be Banach spaces and $A\colon \mathcal{D}(A) \subset E \to F$ be a linear, closed, densely defined, and invertible unbounded operator. Show that if $B \in \mathcal{L}(E, F)$ such that $\|A^{-1}B\|_{\mathcal{L}(E)} < 1$, then $A + B$ is invertible with

   $$(A + B)^{-1} = (I + A^{-1}B)^{-1}\, A^{-1}\, .$$

   Note: Use Neumann series. We sketched the proof in class.

   (b) Now take $F = E$. Use part (a) to show that the *resolvent map*

   $$\lambda \mapsto (A - \lambda I)^{-1}$$

   is a continuous map from $\rho(A) \subset \mathbb{C} \to \mathcal{L}(E)$.

(c) Show further that the resolvent map is differentiable (hence analytic) on $\rho(A)$.
Hint: Note that

$$\frac{(A-\lambda I)^{-1}(A-\lambda_0 I)^{-1}}{\lambda-\lambda_0} = (A-\lambda_0 I)^{-1}\,(A-\lambda I)^{-1}\,.$$

(Why?)

4. Let E, F be Banach spaces and

$$A\colon \mathcal{D}(A) \subset E \to F\,,$$
$$A^\dagger\colon \mathcal{D}(A^\dagger) \subset F^* \to E^*$$

be two linear, closed, densely defined, unbounded operators such that

$$\langle f, Au\rangle_{F^*,F} = \langle A^\dagger f, u\rangle_{E^*,E}$$

for all $f \in \mathcal{D}(A^\dagger)$ and $u \in \mathcal{D}(A)$.
Now suppose there exist

$$G\colon F \to \mathcal{D}(A) \subset E\,,$$
$$G^\dagger\colon E^* \to \mathcal{D}(A^\dagger) \subset F^*$$

such that

$$AG = I_F \qquad \text{and} \qquad A^\dagger G^\dagger = I_{E^*}\,. \tag{*}$$

Then $A^\dagger = A^*$.
Hint: The issue here is whether $\mathcal{D}(A^\dagger) = \mathcal{D}(A^*)$. Note that $f \in \mathcal{D}(A^*)$ if and only if there exists $g \in E^*$ such that

$$\langle f, Au\rangle_{F^*,F} = \langle g, u\rangle_{E^*,E}\,.$$

Note: The above criterion is useful because, on the one hand, $\mathcal{D}(A^*)$ is often difficult to establish directly. On the other hand, the "formal adjoint" $A^\dagger$ and the respective inverses can often be obtained by explicit computation. So the proof that the "natural domain" of the formal adjoint, $\mathcal{D}(A^\dagger)$, already exhausts all of $\mathcal{D}(A^*)$ reduces to the verification of the two identities (*). See next question.

5. Consider $A = -\partial_{xx}$ on $L^2([0,1])$.

   (a) Show that A is self-adjoint with

$$\mathcal{D}(A) = \{u \in L^2([0,1])\colon \partial_{xx}u \in L^2([0,1]), u(0) = u(1) = 0\}\,.$$

   (b) Show that A is symmetric (L^* is an extension of L), but not self-adjoint with

$$\mathcal{D}(A) = \{u \in L^2([0,1])\colon \partial_{xx}u \in L^2([0,1]), u(0) = \partial_x u(0) = u(1) = \partial_x u(1) = 0\}\,.$$

Describe the point, continuous, and residual spectrum in each case.