Functional Analysis

Homework 8

due May 4, 2009

1. Let $\Omega \subset \mathbb{R}^n$ be open and $K\colon L^2(\Omega) \to L^2(\Omega)$ a Hilbert–Schmidt operator, i.e.

$$Kf(x) = \int_\Omega k(x,y)\, f(y)\, \mathrm{d}y$$

with $k \in L^2(\Omega \times \Omega)$. Recall that

$$\|K\|_{\mathrm{HS}}^2 = \sum_{j\in\mathbb{N}} \|Ke_j\|^2$$

for any orthonormal basis $\{e_j\}$ of $L^2(\Omega)$.
$A\colon L^2(\Omega) \to L^2(\Omega)$ is of *trace class*, if

$$\sum_{j\in\mathbb{N}} |\langle Ae_j, e_j\rangle| < \infty\,.$$

(a) Show that the composition of two Hilbert–Schmidt operators is of trace class.

(b) Assume, in addition, that $k(x,y) = \overline{k(y,x)}$. Show that K is self-adjoint and its eigenvalues λ_j satisfy

$$\sum_{j\in\mathbb{N}} |\lambda_j|^2 < \infty\,.$$

2. Let

$$V^* \supset H^* \equiv H \supset V$$

be a Gelfand-triple of Hilbert spaces, with V dense in H and the injection $i\colon V \to H$ continuous. Let $a\colon V \times V \to \mathbb{R}$ be a continuous, coercive bilinear form, define $A\colon V \to V^*$ by $\langle Au, v\rangle_{V^*,V} = a(u,v)$ for all $u, v \in V$, and set

$$\mathcal{D}(A) = \{u \in V\colon Au \in H\}\,.$$

(a) Show that $A\colon \mathcal{D}(A) \to H$ is closed.

(b) Show that $\|Au\|_H$ defines a norm on $\mathcal{D}(A)$ which is equivalent to the graph norm

$$\Big(\|Au\|_H^2 + \|u\|_H^2\Big)^{\frac{1}{2}}\,.$$

(c) Show that $\mathcal{D}(A)$ is a Hilbert space with any of the norms from (ii) and that $A\colon \mathcal{D}(A) \to H$ is an isomorphism.

3. Recall that $L^2(\mathbb{T})$ can be endowed with the norm

$$\|u\|^2 = \sum_{k\in\mathbb{Z}} |u_k|^2 \,,$$

where u_k are the Fourier coefficients of u, i.e.,

$$u(x) = \sum_{k\in\mathbb{Z}} u_k \, \mathrm{e}^{\mathrm{i}kx} \,.$$

Let

$$H^1(\mathbb{T}) = \{u \in L^2(\mathbb{T})\colon u' \in L^2(\mathbb{T})\} \,,$$

endowed with norm

$$\|u\|_1^2 = \sum_{k\in\mathbb{Z}} (1+k^2) \, |u_k|^2 \,.$$

Show that the embedding $L^2 \supset H^1$ is compact.