Functional Analysis

Homework 7

due April 24, 2009

1. Show that a closed subset A of a complete metric space X is compact if and only if for every $\varepsilon > 0$ there exist finitely many $x_1, \ldots, x_k \in X$ such that
$$A \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon)\,.$$

2. Show that the composition (in any order) of a compact and a bounded operator is compact.

3. Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$. Prove the following.
   (a) If T is compact and $u_n \rightharpoonup u$ weakly in E, then $Tu_n \to Tu$ strongly in F.
   (b) If E is reflexive and $u_n \rightharpoonup u$ weakly in E implies that $Tu_n \to Tu$ strongly in F, then T is compact.

4. Show that $T \in \mathcal{K}(E, F)$ if and only if $T^* \in \mathcal{K}(F^*, E^*)$.

   Note: The proof is essentially given on p. 76 in the lecture notes. Write out a *self-contained* version filling in all the gaps. In particular, verify that the assumptions of the Arzelà-Ascoli theorem are satisfied, that $\mathcal{D}(T*) = F*$, and explain the items marked "why?".