Functional Analysis

Homework 2

due February 20, 2009
(submit by 17:00 in Research I, Room 40)

1. (From Folland, p. 124.) Let X and Y be topological spaces.
   (a) If X is connected (see Homework 1) and $f \in C(X, Y)$, then $f(X)$ is connected.
   (b) X is called *pathwise connected* (or *arcwise connected*) if for all $x_0, x_1 \in X$ there exists $f \in C([0,1], X)$ with $f(0) = x_0$ and $f(1) = x_1$. Show that every pathwise connected space is connected.
   (c) Let $X = \{(s,t) \in \mathbb{R}^2 \colon t = \sin(1/s)\} \cup \{(0,0)\}$, with the relative topology induced from $\mathbb{R}^2$. Then X is connected, but not pathwise connected.

2. (From Folland, p. 138.) If $\{X_\alpha\}_{\alpha \in A}$ is a family of topological spaces of which infinitely many are noncompact, then every closed compact subset of $\prod_{\alpha \in A} X_\alpha$ is nowhere dense.

   (Recall that a subset S of a topological space is called *nowhere dense* if $\operatorname{int}(\overline{S}) = \emptyset$.)

3. (From Folland, p. 138.) Let $K \in C([0,1]^2)$. For $f \in C([0,1])$, let
$$Tf(x) = \int_0^1 K(x,y)\, f(y)\, \mathrm{d}y\,.$$
   Then $Tf \in C([0,1])$, and
$$\{Tf \colon \|f\|_u \le 1\}$$
   is precompact in $C([0,1])$.

   (A subset S of a topological space is called *precompact* (or *relatively compact*) if its closure is compact.)

4. (From Folland, p. 138.) Let (X, ρ) be a metric space. A function $f \in C(X)$ is called *Hölder continuous of exponent* $\alpha > 0$ if the quantity
$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha}$$
   is finite. If X is compact, prove that
$$\{f \in C(X) \colon \|f\|_u \le 1 \text{ and } N_\alpha(f) \le 1\}$$
   is compact in $C(X)$.