Complex Analysis

Homework 9

Due in class Thursday, November 25, 2021

1. (Stein & Shakarchi, Chapter 8 Exercise 12.) A complex number $w \in \mathbb{D}$ is a fixed point for the map $f\colon \mathbb{D} \to \mathbb{D}$ if $f(w) = w$.

   (a) Prove that if $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic and has two distinct fixed points, then f is the identity.

   Hint: Consider the function $g = \psi_w \circ f \circ \psi_w$ where w is one of the fixed points and ψ_w is the automorphism of the disc that exchanges 0 and w.

   (b) Must every holomorphic function $f\colon \mathbb{D} \to \mathbb{D}$ have a fixed point?

   Hint: Consider the upper half-plane.

2. We say that a power series with coefficients a_n is *asymptotic* to a function $f\colon \mathbb{R} \to \mathbb{R}$, written

$$f(x) \sim \sum_{n=0}^{\infty} a_n\, x^n \,,$$

   if, for every $N \in \mathbb{N}$,

$$f(x) - \sum_{n=0}^{N} a_n\, x^n = O(x^{N+1}) \quad \text{as } x \searrow 0 \,. \tag{*}$$

   Notes: An asymptotic series may not be convergent. The concept carries over to functions on $\mathbb{C}$; in that case, the validity of an asymptotic expansions is typically restricted to a sector of the complex plane.

   (a) Show that

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n \quad \text{as } x \searrow 0$$

   and

$$\frac{1}{1-x} + \mathrm{e}^{-1/x}\, \sin(\mathrm{e}^{1/x}) \sim \sum_{n=0}^{\infty} x^n \quad \text{as } x \searrow 0 \,.$$

   In other words, an asymptotic series does not uniquely characterize a function.

(b) Demonstrate that term-by-term differentiation of the asymptotic series, in the second case, does not yield an asymptotic series for the derivative of the left-hand function.

3. Let

$$I(s) = \int_0^b \mathrm{e}^{-sx} f(x) \, \mathrm{d}x$$

where $b > 0$ and $f \colon [0,1] \to \mathbb{R}$ is continuous and has an asymptotic expansion of the form (*). Prove that

$$I(s) \sim \sum_{n=0}^{\infty} a_n \, \Gamma(n+1) \, s^{-n-1} \quad \text{as } s \to \infty \, .$$

(This is a simplified version of what is known as *Watson's lemma*.)