Complex Analysis

Homework 7

Due in class Thursday, November 11, 2021

1. (Cf. Stein & Shakarchi, Chapter 2 Exercise 12.)

   Suppose that $u\colon \mathbb{D} \to \mathbb{R}$ is harmonic, i.e., u is twice continuously differentiable as a function of two real variables and satisfies

   $$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,.$$

   Show that there exists a holomorphic function f on $\mathbb{D}$ such that $u = \operatorname{Re} f$ and $\operatorname{Im} f$ is uniquely defined up on an additive real constant.

   Hint: Show that if $f = u + \mathrm{i}v$ were holomorphic, then $f'(z) = 2\,\partial u/\partial z$. Thus, if you can show that $\partial u/\partial z$ is holomorphic, then you can recover f via its primitive (explain in detail!).

2. (Stein & Shakarchi, Chapter 8 Exercise 1.) A holomorphic mapping $f\colon U \to V$ is a local bijection on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z, so that $f\colon D \to f(D)$ is a bijection. Prove that a holomorphic map $f\colon U \to V$ is a local bijection on U if and only if $f'(z) = 0$ for all $z \in U$.

   Hint: Modify the Rouché theorem argument shown in class.

3. (Stein & Shakarchi, Chapter 8 Exercise 2.) Supppose $f(z)$ is holomorphic near $z = 0$ and $f(0) = f'(0) = 0$, while $f''(0) \neq 0$. Show that there are two curves γ_1 and γ_2 that pass through the origin, are orthogonal at the origin, and so that f restricted to γ_1 is real and has a minimum at 0, while f restricted to γ_2 is also real but has a maximum at 0.

   Hint: Write $f(z) = (g(z))^2$ for z near 0, and consider the mapping $z \mapsto g(z)$ and its inverse.

4. (Stein & Shakarchi, Chapter 8 Exercise 5.) Prove that $f(z) = -\frac{1}{2}\,(z + 1/z)$ is a conformal map from the half-disc $\{z = x + \mathrm{i}y\colon |z| < 1, y > 0\}$ to the upper half-plane.

   Hint: The equation $f(z) = w$ reduces to the quadratic equation $z^2 + 2wz + 1 = 0$, which has two distinct roots in $\mathbb{C}$ whenever $w = \pm 1$. This is certainly the case if $w \in \mathbb{H}$.