Complex Analysis

Homework 4

Due in class Thursday, October 14, 2021

1. (Stein & Shakarchi, Chapter 3 Exercise 2.) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,\mathrm{d}x\,.$$

2. (Stein & Shakarchi, Chapter 3 Exercise 4.) Show that, for $a > 0$,

$$\int_{-\infty}^{\infty} \frac{x\,\sin x}{x^2+a^2}\,\mathrm{d}x = \pi\,\mathrm{e}^{-a}\,.$$

3. (Stein & Shakarchi, Chapter 3 Exercise 6.) Show that, for $n \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,\mathrm{d}x = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,.$$

4. (Stein & Shakarchi, Chapter 3 Exercise 16.) Suppose f and g are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_\varepsilon(z) = f(z) + \varepsilon\,g(z)\,.$$

Show that, if ε is sufficiently small, then

(a) f_ε has a unique zero in $|z| \leq 1$, and

(b) if z_ε denotes this zero, the mapping $\varepsilon \mapsto z_\varepsilon$ is continuous.