1. (a) Find all solutions to the equation

$$z^5 = 1 - \mathrm{i}\,.$$

(b) List all possible values for i^{i}. Which value corresponds to the principal branch of the logarithm?

(5+5)

(a) $z^5 = \sqrt{2}\, e^{(-\frac{1}{4} + 2n)\pi i}$ $\quad n \in \mathbb{N}$

$\Rightarrow z = 2^{\frac{1}{10}}\, e^{(-\frac{1}{20} + \frac{2}{5}n)\pi i}$

with distinct values e.g. for $n = 0, 1, 2, 3, 4$

(b) $i^i = e^{i \log i} = e^{i(\frac{\pi}{2} i + 2n\pi i)}$ $\quad n \in \mathbb{N}$

$= e^{-\frac{\pi}{2} - 2n\pi}$

where $n=0$ corresponds to the principal branch of $\log i$.

2. What is the radius of convergence of the power series of

$$f(z) = \frac{(z-1)^2}{z^2-1}$$

about the point $z_0 = 1 + i$? (5)

$$f(z) = \frac{(z-1)^2}{(z-1)(z+1)} = \frac{z-1}{z+1}$$

$\Rightarrow$ f is meromorphic with a single pole at $z=-1$

$\Rightarrow$ the radius of convergence is

$$|z_0 - (-1)| = |2+i| = \sqrt{4+1} = \sqrt{5}$$

<u>Note</u>: there is no need to compute the power series, but in this case it's easily done:

$$\frac{z-1}{z+1} = \frac{z-(1+i)+i}{z-(1+i)+2+i}$$

$$= \left((z-z_0)+i\right)\frac{1}{2+i}\underbrace{\frac{1}{1-\frac{z-z_0}{-2-i}}}_{}$$

$$\overset{(*)}{=} 1 - \frac{z-z_0}{2+i} + \frac{(z-z_0)^2}{(2+i)^2} - \cdots$$

The geometric series $(*)$ converges for

$$\left|\frac{z-z_0}{2+i}\right| < 1$$

$\Rightarrow$ $|z-z_0| < \sqrt{5}$ which confirms the earlier computation

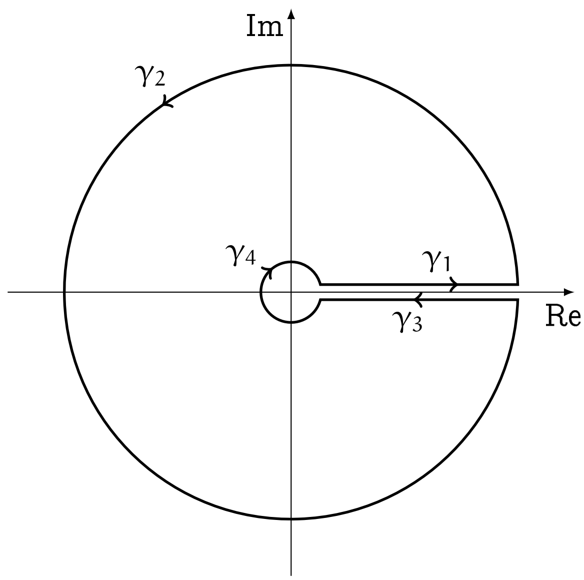
3. Integrate the square root function

$$f(z) = \sqrt{z}$$

around the unit circle $C = \{z\colon |z| = 1\}$ with standard orientation in two different ways:

(a) by parameterizing the unit circle in the standard way,

(b) by integrating around a contour as sketched below.



(5+10)

(a) $$\int_C f\,dz = \int_C e^{\frac{1}{2}\log z}\,dz$$

Taking the branch cut as depicted for (b), we have $0 < \arg z < 2\pi$:

$$\int_C f\,dz = \int_0^{2\pi} e^{\frac{1}{2}(i\theta)}\, i e^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} e^{\frac{3}{2}i\theta}\, d\theta = \frac{i}{\frac{3}{2}i}\, e^{\frac{3}{2}i\theta}\Big|_0^{2\pi} = \frac{2}{3}\left(e^{3\pi i} - e^{0}\right)$$

$$= -\frac{4}{3}$$

(b) Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_4$. By CT:

$$\int_\gamma f \, dz = 0$$

Let $\gamma_4 = C_\varepsilon(0)$. Since f is bounded near 0,

$$\left| \int_{C_\varepsilon(0)} f \, dz \right| \xrightarrow[\varepsilon \to 0]{} 0$$

$$\int_{\gamma_1} f \, dz \xrightarrow[\varepsilon \to 0]{} \int_0^1 x^{\frac{1}{2}} \, dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}$$

$$\int_{\gamma_3} f \, dz \xrightarrow[\varepsilon \to 0]{} \int_1^0 \underbrace{e^{\frac{1}{2}(\log x + 2\pi i)}}_{= x^{\frac{1}{2}} e^{\pi i} = -x^{\frac{1}{2}}} \, dx = \int_0^1 x^{\frac{1}{2}} \, dx = \frac{2}{3}$$

Altogether, as $\varepsilon \to 0$, noting that $C = \gamma_2$:

$$\int_C f \, dz + \int_{\gamma_1} f \, dz + \int_{\gamma_3} f \, dz = 0$$

$$\Rightarrow \int_C f \, dz = -\frac{4}{3} \qquad \text{as in (a).}$$

4. Find

$$\int_C \frac{1}{z^2\,(z+1)^2}\,dz$$

where $C = \{z \colon |z| = \frac{1}{2}\}$ with standard orientation. (10)

The function $f(z) = \dfrac{1}{z^2\,(z+1)^2}$ has two poles of order 2 at $z=0,\ z=-1$.

Only the pole at the origin is within the contour of integration.

$\Rightarrow \displaystyle\int_C f\,dz = 2\pi i \operatorname{Res}(f,0)$ by the residue theorem

To compute the residue, write

$$\frac{1}{1+z} = 1 - z + z^2 - \text{h.o.t.}$$

$$\Rightarrow \frac{1}{(1+z)^2} = 1 - 2z + 3z^2 + \text{h.o.t.}$$

$$\Rightarrow \frac{1}{z^2}\,\frac{1}{(1+z)^2} = \frac{1}{z^2} - \frac{2}{z} + 3 + \dots$$

$$\Rightarrow \operatorname{Res}(f,0) = -2$$

$$\Rightarrow \int_C f\,dz = -4\pi i$$

5. Suppose $f(z)$ is entire and suppose there exist $c > 0$ and $n \in \mathbb{N}$ such that

$$|f(z)| \leq c\,|z|^n$$

for all $z \in \mathbb{C}$. Prove that f is a polynomial.

Hint: Consider $f^{(m)}$ for $m > n$. (10)

Solution 1: Use the CIF:

$$f^{(m)}(0) = \frac{m!}{2\pi i} \int_{C_R(0)} \frac{f(\zeta)}{(\zeta - 0)^{m+1}}\, d\zeta$$

$$\Rightarrow |f^{(m)}(0)| \leq \frac{m!}{2\pi} \int_0^{2\pi} \frac{c\, R^n}{R^{m+1}}\, R\, d\theta \qquad \zeta = Re^{i\theta}$$

$$= \tilde{c}\, R^{n-m} \xrightarrow[R\to\infty]{} 0 \qquad \text{if } m > n$$

$$\Rightarrow f^{(m)}(0) = 0 \quad \text{for } m > n$$

So the Taylor series of f has non-zero coefficients up to at most order n, so it is a polynomial.

Solution 2: $g(z) = z^{-n} f(z)$ is bounded near ∞

$\Rightarrow$ $h(z) = g(\frac{1}{z})$ has a removable singularity at 0,

i.e., it has a local power series expansion

$$h(z) = \sum_{j=0}^{\infty} a_j z^j \quad \Rightarrow \quad f(z) = z^n \sum_{j=0}^{\infty} a_j z^{-j}$$

Since f is entire, $a_j = 0$ for $j > n$.