# Complex Analysis

## Final Exam

## December 20, 2021

1. (a) Find the real and imaginary parts of $\dfrac{z+2}{z-1}$.
   (b) Find all solutions to the equation $z^4 = 1$.
   (c) Find all complex solutions to the equation $\cos z = \sin z$.
   *Hint:* Let $t = e^{iz}$.
   (d) Find the radius of convergence of $\dfrac{z^3-1}{z^2+3z-4}$ about the point $z = 0$.

(5+5+5+5)

2. Short problems (I):
   (a) Is it true that $|a^b| = |a|^{|b|}$?
   (b) Suppose $f$ is entire with $|f| \geq 1$ on all of $\mathbb{C}$. Show that $f$ is a constant.
   (c) The function $f(z) = z^{-2}$ goes to zero as $|z| \to \infty$. Does this contradict Liouville's theorem?

(5+5+5)

3. Short problems (II):
   (a) Show that $f(z) = 4z^6 + z^2 - 2$ has all of its zeros inside the unit disc.
   (b) Is
   $$f(z) = \frac{1}{1+z^2}$$
   holomorphic on the exterior of the unit disc, $\{z\colon |z| > 1\}$?
   (c) Let $f\colon \mathbb{D} \to \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc, be holomorphic with $f(0) = 0$. Show that the power series expansion of
   $$g(z) = \frac{1}{1 - 2\,f(z)}$$
   converges on $\{z\colon |z| < \frac{1}{2}\}$.

(5+5+5)

4. (a) Integrate the function

$$f(z) = \operatorname{Re} z$$

around the boundary of the square $Q = [0,1] + i[0,1]$ with standard orientation.

(b) Is $f$ holomorphic?

(10+5)

5. Let $\gamma = \{\zeta \colon |\zeta| = 1, \operatorname{Im} \zeta \geq 0\}$ denote the upper-half unit circle endowed with standard orientation. Show that, for $|z| < 1$,

$$g(z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} \, d\zeta$$

is given by

$$g(z) = \frac{1}{2} + \frac{1}{2\pi i} \log \frac{1+z}{1-z} \, .$$

*Note:* Be careful and explicit about the required branch cut. (10)

6. Use contour integration to compute

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} \, dx \, .$$

(10)

7. Let $\Omega \subset \mathbb{C}$ be open and simply connected. Show that for any two $u, w \in \Omega$ there exists a conformal map $f \colon \Omega \to \Omega$ such that $f(u) = w$. (5)

8. Suppose that $f$ is holomorphic and of moderate decay on the strip $|\operatorname{Im} z| < a$, i.e.,

$$|f(x + iy)| \leq \frac{A}{1 + x^2}$$

for some $A > 0$ and all $x \in \mathbb{R}$ and $|y| < a$.

Show that the Fourier transform of $f$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} \, f(x) \, dx$$

satisfies the estimate

$$|\hat{f}(\xi)| \leq B \, e^{-2\pi b \xi}$$

for some $B > 0$, every $0 < b < a$, and $\xi > 0$. (10)