1. (a) Find the real and imaginary parts of $\dfrac{z+2}{z-1}$.

(b) Find all solutions to the equation $z^4 = 1$.

(c) Find all complex solutions to the equation $\cos z = \sin z$.

Hint: Let $\mathrm{t} = \mathrm{e}^{\mathrm{i}z}$.

(d) Find the radius of convergence of $\dfrac{z^3-1}{z^2+3z-4}$ about $z=0$.

(5+5+5+5)

(a) Let $z = x+iy$, so that $\dfrac{z+2}{z-1} = \dfrac{x+iy+2}{x+iy-1} \cdot \dfrac{x-iy-1}{x-iy-1}$

$$= \frac{(x+2)(x-1)+y^2+iy(x-1-x-2)}{(x-1)^2+y^2} = \underbrace{\frac{(x+2)(x-1)+y^2}{(x-1)^2+y^2}}_{\text{real part}} + i\underbrace{\frac{-3y}{(x-1)^2+y^2}}_{\text{imag. part}}$$

(b) $|z|=1 \Rightarrow z = e^{i\theta}$ with $e^{4i\theta} = e^{0}$

$\Rightarrow \theta \in \frac{\pi}{2}\mathbb{Z}$ leading to 4 unique solutions $1, i, -1, -i$

(c) Following the hint: $\frac{1}{2}\left(t+\frac{1}{t}\right) = \frac{1}{2i}\left(t-\frac{1}{t}\right)$

$\Rightarrow it^2 + i = t^2 - 1$

$\Rightarrow t^2 = \dfrac{1+i}{1-i} = \frac{1}{2}(1+2i-1) = i$

$\Rightarrow t = e^{i\frac{\pi}{4}}$ or $t = e^{i\frac{5\pi}{4}}$

$\Rightarrow z = \frac{\pi}{4} + \pi\mathbb{Z}$ (i.e. all solutions are real !)

(d) $z^2 + 3z - 4 = (z-1)(z+4)$

Since 1 is also a root of z^3-1, it is a removable singularity.

The function has a pole at $z=4$, so the radius of convergence is 4.

2. Short problems (I):

(a) Is it true that $|a^b| = |a|^{|b|}$?

(b) Suppose f is entire with $|f| \geq 1$ on all of $\mathbb{C}$. Show that f is a constant.

(c) The function $f(z) = z^{-2}$ goes to zero as $|z| \to \infty$. Does this contradict Liouville's theorem?

(5+5+5)

(a) No. $|e^{i\pi}| = 1 \neq e^{\pi}$

(b) Then $g(z) = \frac{1}{f(z)}$ is also entire and bounded.

By Liouville's theorem, g is a constant, so f is a constant.

(c) f has a pole at $z=0$, so is not entire and Liouville's theorem does not apply.

3. Short problems (II):

(a) Show that $f(z) = 4z^6 + z^2 - 2$ has all of its zeros inside the unit disc.

(b) Is

$$f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

holomorphic on the exterior of the unit disc, $\{z\colon |z| > 1\}$?

(c) Let $f\colon \mathbb{D} \to \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disc, with $f(0) = 0$. Show that the power series expansion of

$$g(z) = \frac{1}{1 - 2f(z)}$$

converges on $\cancel{\mathbb{D}}$. $D_{\frac{1}{2}}(0)$.

(5+5+5)

(a) On the unit circle, $|4z^6| = 4 > |z^2 - 2| = 3$,

so f has the same number of zeros as $4z^6$, namely 6, inside the unit disc by Rouché's theorem. By the fundamental theorem of algebra, these are already all of the zeros.

(b) $$\int_{C_R(0)} f\, dz = 2\pi i \Big(\underbrace{\mathrm{Res}(f, i)}_{\frac{1}{2i}} + \underbrace{\mathrm{Res}(f, -i)}_{= -\frac{1}{2i}} \Big) = 0 \qquad (R > 1)$$

So contour integrals encircling the unit disc vanish, so f is holomorphic on the exterior of the unit disc.

(c) By the Schwarz lemma, f maps $D_{\frac{1}{2}}$ into $D_{\frac{1}{2}}$, so $2f$ maps $D_{\frac{1}{2}}$ into $\mathbb{D}$.

Moreover, the geometric series $z \mapsto \frac{1}{1-z}$ converges on $\mathbb{D}$.

4. (a) Integrate the function

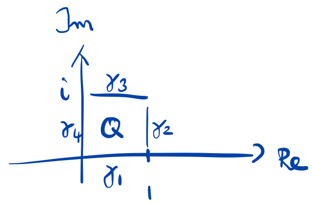
$$f(z) = \operatorname{Re} z$$

around the boundary of the square $Q = [0,1] + i[0,1]$.

(b) Is f holomorphic on Q?

(10+5)

(a) Write $z = x + iy$



$$\int_{\gamma_1} x\,dz = \int_0^1 x\,dx = \frac{1}{2} \qquad \int_{\gamma_3} x\,dz = \int_1^0 x\,dz = -\frac{1}{2}$$

$$\int_{\gamma_2} x\,dz = \int_0^1 1 \cdot i\,dy = i \qquad \int_{\gamma_4} x\,dz = \int_1^0 0 \cdot i\,dy = 0$$

"Jacobian factor"!

$$\Rightarrow \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} \right) \operatorname{Re} z\,dz = i$$

(b) If f were holomorphic, the line integral in (a) must be zero.

So $\operatorname{Re} z$ is not holomorphic.

(Alternatively use the CR-equations.)

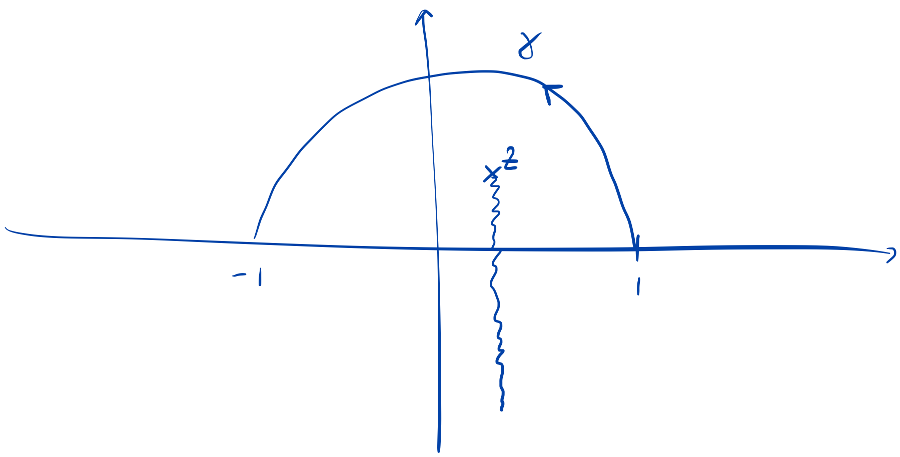
5. Let γ denote the upper-half unit circle in $\mathbb{C}$ with standard orientation. Show that, for $|z| < 1$,

$$g(z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\, d\zeta$$

is given by

$$g(z) = \frac{1}{2} + \frac{1}{2\pi i}\log\frac{1+z}{1-z}.$$

Note: Be careful and explicit about the required branch cut. (10)



We know that the primitive of $\frac{1}{\zeta - z}$ is $\log(\zeta - z)$. We need to choose the branch cut as indicated so that it does not cross γ.

Then

$$g(z) = \frac{1}{2\pi i}\log(\zeta - z)\Big|_{\zeta=1}^{\zeta=-1} \quad \text{by the complex FTC.}$$

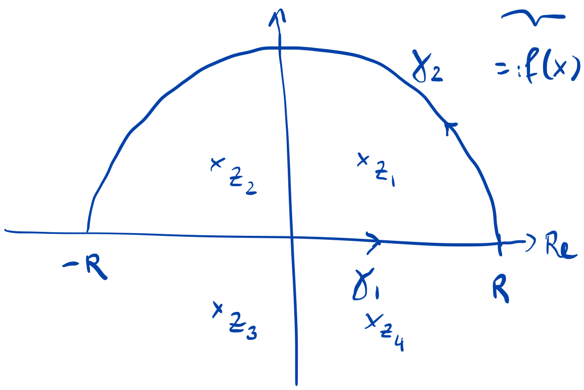
$$= \frac{1}{2\pi i}\Big(\underbrace{\log(-1-z)}_{\log(1+z)+\pi i} - \log(1-z)\Big)$$

$$= \frac{1}{2} + \frac{1}{2\pi i}\Big(\underbrace{\log(1+z) + \log(1-z)}_{=\log\frac{1+z}{1-z}}\Big)$$

because $|\arg(1+z)| < \frac{\pi}{2}$ and $|\arg(1-z)| < \frac{\pi}{2}$ ∇

6. Use contour integration to compute

$$I = \int_{-\infty}^{\infty} \underbrace{\frac{1}{x^4+1}}_{=:f(x)} \, dx. \tag{10}$$



$z^4 = -1$ has roots $z_1 = e^{i\frac{\pi}{4}}$, $z_2 = e^{i\frac{3}{4}\pi}$, $z_3 = e^{i\frac{5}{4}\pi}$, $z_4 = e^{i\frac{7}{4}\pi}$

$$\left| \int_{\gamma_2} \frac{1}{z^4+1} \, dz \right| \leq \pi R \frac{1}{R^4 - 1} \longrightarrow 0 \quad \text{as } R \to \infty$$

So, in the limit $R \to \infty$:

$$I = 2\pi i \left(\operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2) \right)$$

$$= 2\pi i \left(\frac{1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} + \frac{1}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} \right)$$

$$= \frac{2\pi i}{2^{3/2}} \left(\underbrace{\frac{1}{(1+i-(-1+i))((1+i)-(-1-i))((1+i)-(1-i))}}_{= \frac{1}{2 \cdot 2(1+i) \cdot 2i}} + \underbrace{\frac{1}{(-1+i-(1+i))(-1+i-(-1-i))(-1+i-(1-i))}}_{= \frac{1}{-2 \cdot 2i \cdot 2(i-1)}} \right)$$

$$= \frac{\pi}{\sqrt{2}} \underbrace{\left(\frac{1}{1+i} + \frac{1}{1-i} \right)}_{= \frac{1-i+1+i}{2} = 1} \quad \Rightarrow \quad \boxed{I = \frac{\pi}{\sqrt{2}}}$$

7. Let $\Omega \subset \mathbb{C}$ be open and simply connected. Show that for any two $u, w \in \Omega$ there exists a conformal map $f \colon \Omega \to \Omega$ such that $f(u) = w$. (5)

Let F_u be the conformal map $\Omega \to \mathbb{D}$ with $F_u(u) = 0$

guaranteed by the Riemann mapping theorem.

Likewise, F_v the conformal map $\Omega \to \mathbb{D}$ with $F_v(v) = 0$.

Then $f = F_v^{-1} \circ F_u$ has the requested properties.

Note: This is the version of the Riemann mapping theorem from class.

Sometimes, it is stated as simply guaranteeing the existence of

a conformal map $F: \Omega \to \mathbb{D}$

Then use an automorphism of $\mathbb{D}$ to map $F(u)$ to $F(v)$, φ say, and set

$$f = F^{-1} \circ \varphi \circ F.$$

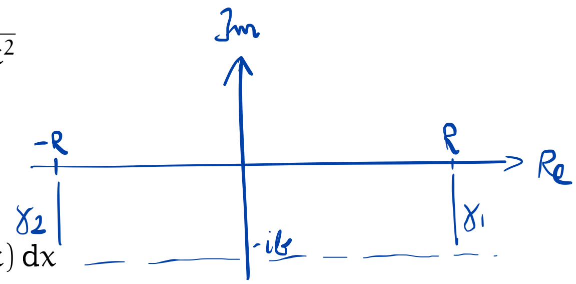
8. Suppose that f is holomorphic and of moderate decay on the strip $|\operatorname{Im} z| < a$, i.e.,

$$|f(x+iy)| \leq \frac{A}{1+x^2}$$

for some $A > 0$ and all $x \in \mathbb{R}$ and $|y| < a$.

Show that the Fourier transform of f,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\,dx$$



satisfies the estimate

$$|\hat{f}(\xi)| \leq B\,e^{-2\pi b \xi}$$

for some $B > 0$, every $0 < b < a$, and $\xi > 0$. (10)

As usual, shift the contour of integration to $z = x - ib$, $x \in \mathbb{R}$:

Since $\left| \int_0^b \underbrace{e^{-2\pi i (R-iy)\xi}}_{|\dots| \leq 1} \underbrace{f(R-iy)}_{|\dots| \leq \frac{A}{1+R^2}}\, i\,dy \right| \leq b \frac{A}{1+R^2} \longrightarrow 0$ as $R \to \infty$,

the contributions from γ_1 and γ_2 vanish as $R \to \infty$, so that, for $\xi > 0$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i (x-ib)\xi} f(x-ib)\,dx$$

$$\Rightarrow |\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} e^{-2\pi b \xi} \frac{A}{1+x^2}\,dx$$

$$= B\,e^{-2\pi b \xi} \quad \text{where } B = A \underbrace{\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx}_{=\pi} < \infty$$