# Midterm Solutions

1. (a) $$\lim_{s\to -1} \frac{\frac{1}{s}-1}{s^3-1} = \frac{-2}{-2} = 1.$$

Note: no canceling of common factors required!

$$\left[\text{But: } \lim_{s\to 1} \frac{\frac{1}{s}-1}{s^3-1} = \lim_{s\to 1} \frac{1}{s}\,\frac{1-s}{(s-1)(s^2+s+1)} = \lim_{s\to 1} \frac{-1}{s(s^2+s+1)} = -\frac{1}{3}\right]$$

(b) $$\lim_{x\to\infty} \frac{e^{2x}+x^3+\ln x}{3e^{2x}-x^3+\cos x} = \lim_{x\to\infty} \frac{1+\frac{x^3}{e^{2x}}+\frac{\ln x}{e^{2x}}}{3-\frac{x^3}{e^{2x}}+\frac{\cos x}{e^{2x}}} = \frac{1}{3}$$

(c) $$\lim_{r\nearrow 1} \frac{|r-1|}{r^2+1} = \lim_{r\nearrow 1} \frac{|r-1|}{(r+1)(r-1)} = -\frac{1}{2}$$

Likewise, $$\lim_{r\searrow 1} \frac{|r-1|}{r^2+1} = \frac{1}{2}$$

so $\displaystyle\lim_{r\to 1} \frac{|r+1|}{r^2+1}$ does not exist.

2. The two functions $y = 4-x$ and $y = x^2+2$ touch at $x=1$ and are continuous. Thus, by the squeeze law,

$$\lim_{x\to 1} f(x) = 3 = f(1),$$

so $f$ is continuous at $x=1$.

3. $f(x) = x^7-3x-1$ is continuous, $f(-1) = -1+3-1 = 1 > 0$, and $f(1) = 1-3-1 = -3 < 0$. So, by the intermediate value theorem, $f$ must have at least one root in the interval $(-1,1)$.

4. (a) $y = \arctan x \quad \Rightarrow \quad x = \tan y$

Now differentiate with respect to $x$: $\quad 1 = \tan' y \dfrac{dy}{dx}$

Moreover, $\tan' y = \dfrac{\cos^2 y - (-\sin y)\sin y}{\cos^2 y} = \dfrac{1}{\cos^2 y}$

$$x^2 = \tan^2 y = \frac{\sin^2 y}{\cos^2 y} = \frac{1-\cos^2 y}{\cos^2 y} \Rightarrow \cos^2 y \, x^2 = 1 - \cos^2 y$$
$$\Rightarrow \cos^2 y = \frac{1}{1+x^2}$$

Putting everything together: $\dfrac{dy}{dx} = \dfrac{1}{\tan' y} = \cos^2 y = \dfrac{1}{1+x^2}$.

(b) • $D(f) = \mathbb{R}$, so no vertical asymptotes

• $\lim\limits_{x\to\infty} (2\arctan x - x) = -\infty$, $\lim\limits_{x\to-\infty} (2\arctan x - x) = \infty$

$\Rightarrow$ no horizontal asymptotes

• $f'(x) = \dfrac{2}{1+x^2} - 1 = \dfrac{1-x^2}{1+x^2} = \dfrac{(1-x)(1+x)}{1+x^2}$

$\Rightarrow$ for $x < -1$, $f'(x) < 0 \Rightarrow f$ is decreasing

for $x \in (-1,1)$, $f'(x) > 0 \Rightarrow f$ is increasing

for $x > 1$, $f''(x) < 0 \Rightarrow f$ is decreasing

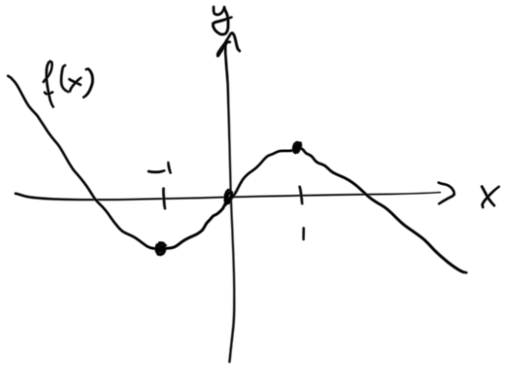
So $f$ has a local min at $(-1, f(-1))$, a local max at $(1, f(1))$.

• $f''(x) = \dfrac{-4x}{(1+x^2)^2}$

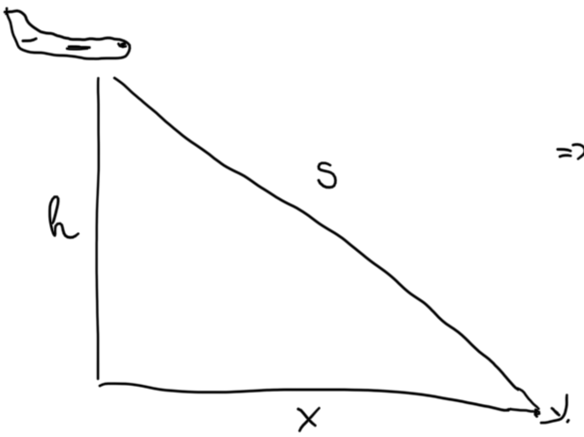
$\Rightarrow$ for $x<0$, $f''(x) > 0$, so $f$ is concave up

for $x > 0$, $f''(x) < 0$ so $f$ is concave down for $x>0$, $f''(x) < 0$, so $f$ is concave down.

So $f$ has a point of inflection at $x=0$




5.




$$s^2 = h^2 + x^2$$

$$\Rightarrow 2s \frac{ds}{dt} = 2x \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$$

Here: $s = 10$ km, $h = 6$ km, $\frac{ds}{dt} = 400 \frac{\text{km}}{\text{h}}$

$$\Rightarrow x^2 = 100 \text{ km}^2 - 36 \text{ km}^2 = 64 \text{ km}^2$$

$$\Rightarrow x = 8 \text{ km}$$

$$\Rightarrow \frac{dx}{dt} = \frac{10 \text{ km}}{8 \text{ km}} \cdot 400 \frac{\text{km}}{\text{h}} = 500 \frac{\text{km}}{\text{h}}$$

6. (a) Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx$

$$\Rightarrow \int \frac{1}{x^3} e^{\frac{1}{x}} dx = -\int u e^u du$$

$$= -ue^u + \int e^u du$$

$$= -u e^{u} + e^{u} + C$$

$$= -\frac{e^{\frac{1}{x}}}{x} + e^{\frac{1}{x}} + C$$

(b) Need partial fractions:

$$\frac{x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2}{x^2(x^2+1)}$$

So we need to solve

$$x+1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

Collect different powers of $x$:

$x^0$ : $1 = B$

$x^1$ : $1 = A$

$x^2$ : $0 = B + D$ $\Rightarrow$ $D = -B = -1$

$x^3$ : $0 = A + C$ $\Rightarrow$ $C = -A = -1$

$$\Rightarrow \int \frac{x+1}{x^2(x^2+1)}\,dx = \underbrace{\int \frac{1}{x}\,dx}_{=\ln|x|} + \underbrace{\int \frac{1}{x^2}\,dx}_{=-\frac{1}{x}} - \underbrace{\int \frac{x}{x^2+1}\,dx}_{=\frac{1}{2}\int \frac{du}{u}} - \underbrace{\int \frac{1}{x^2+1}\,dx}_{=\arctan x}$$

$u = x^2+1$
$\Rightarrow du = 2x\,dx$

$$= \frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln|x^2+1|$$

$$= \ln|x| - \frac{1}{x} - \frac{1}{2}\ln|x^2+1| - \arctan x + C$$

(c) The integral is zero as both terms are periodic with period $\pi$ and shifted copies of each other, so they cancel out.

7. Let $G(x) = \int_0^x \frac{e^t}{t}\,dt \quad \Rightarrow \quad G'(x) = \frac{e^x}{x}$

$$\Rightarrow F(x) = G(x) - G(\sqrt{x})$$

$$\Rightarrow F'(x) = G'(x) - G'(\sqrt{x})\,\frac{1}{2}x^{-\frac{1}{2}} \qquad \text{(chain rule!)}$$

$$= \frac{e^x}{x} - \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\frac{1}{2\sqrt{x}}$$

$$= \frac{e^x - \frac{1}{2}e^{\sqrt{x}}}{x}$$