Homework 9 Solutions

1. Let $N(t) = N_0 e^{-rt}$ denote the number of ^{14}C atoms in the sample.

$$\tfrac{1}{2} N_0 = N_0 e^{-r t_{1/2}} \quad \text{where } t_{1/2} = 5730\text{ a is the half-life of } {}^{14}C.$$

$$0.74\, N_0 = N_0 e^{-rT} \quad \text{where } T \text{ is the age of the sample}$$

$$\Rightarrow \left.\begin{aligned} \ln 2 &= r\, t_{1/2} \\ \ln \tfrac{1}{0.74} &= rT \end{aligned}\right\} \quad \frac{\ln 2}{\ln \frac{1}{0.74}} = \frac{t_{1/2}}{T}$$

$$\Rightarrow T = t_{1/2} \frac{\ln \frac{1}{0.74}}{\ln 2} \approx 2430 \text{ a}$$

2. (a) $$m \frac{dv}{dt} = -kv$$

$$\Rightarrow \frac{dv}{dt} = -\frac{k}{m} v \quad \Rightarrow \quad v(t) = v_0 e^{-\frac{k}{m}t}$$

So the velocity is decreasing exponentially, but it does not reach zero in a finite time.

(b) $$\frac{dx}{dt} = v = v_0 e^{-\frac{k}{m}t}$$

Let x_0 be the initial position and x_{final} the final position as $t \to \infty$.

Then
$$x_{final} = x_0 + \int_0^\infty v_0 e^{-\frac{k}{m}t}\, dt = x_0 - v_0 \frac{m}{k} e^{-\frac{k}{m}t} \Big|_0^\infty$$

$$= x_0 + v_0 \frac{m}{k}$$

So the particle travels a finite distance in infinite time.

(c) Since $m\frac{dv}{dt} = F$ and $\frac{dx}{dt} = v$, $m\frac{dv}{dx} = \frac{F}{v}$ $\Rightarrow F\,dx = mv\,dv$

$$\Rightarrow W = \int_{x_0}^{x_{final}} F\,dx = m\int_{v_0}^{0} v\,dv = -\frac{1}{2}mv_0^2$$

I.e., the particle loses its entire initial kinetic energy.

Note: this computation is true in general so long as Newton's second law applies – it does not depend on $F = -kv$!

3. $\frac{dy}{dx} = y(2x+1)$, $y(0) = 2$

$$\Rightarrow \int_2^{y(x)} \frac{dy}{y} = \int_0^x (2x+1)\,dx \Rightarrow \ln y \Big|_2^{y(x)} = x^2+x \Big|_0^x$$

$$\Rightarrow \ln\frac{y(x)}{2} = x^2+x$$

$$\Rightarrow y(x) = 2e^{x^2+x}$$

4. $e^x e^y \frac{dy}{dx} + e^x e^y = x^2$ $\qquad y(0) = 0$

$$\Rightarrow \frac{d}{dx}\left(e^x e^y\right) = x^2$$

$$\Rightarrow \int_0^x \frac{d}{dx}\left(e^x e^y\right)dx = \int_0^x x^2\,dx$$

$$\Rightarrow e^x e^{y(x)} - \underbrace{e^0 e^{y(0)}}_{=1} = \frac{1}{3}x^3$$

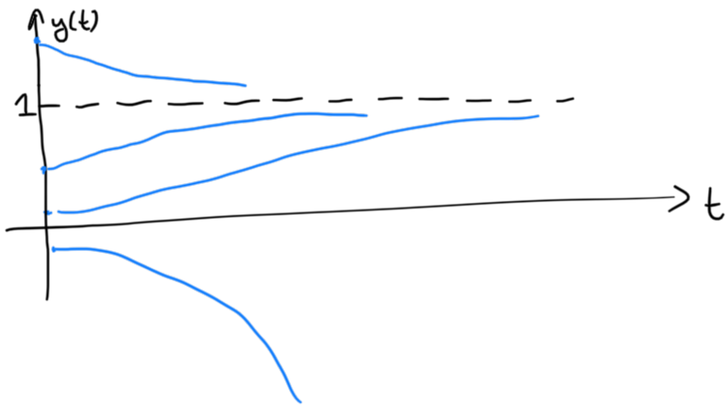
$$\Rightarrow e^{x+y(x)} = \frac{1}{3}x^3 + 1$$

$$\Rightarrow \quad y(x) = \ln\left(\tfrac{1}{3}x^3+1\right) - x$$

5.(a) $\frac{dy}{dx} = y - y^2 = y(1-y)$

- 0 for $y=0$, $y=1$
- <0 for $y<0$, $y>1$
- >0 for $y\in(0,1)$

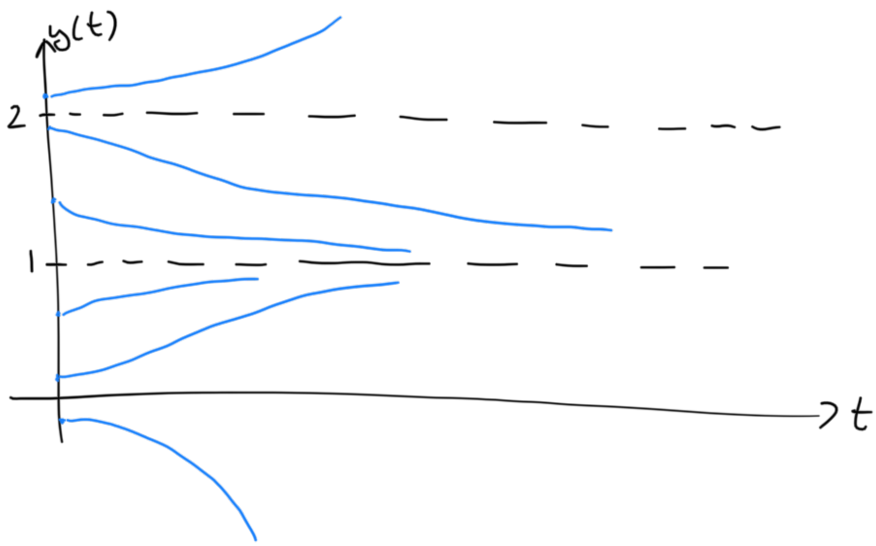
$y=0$ is an unstable, $y=1$ a stable equilibrium



(b) $\frac{dy}{dx} = y(y-1)(y-2)$

- 0 for $y=0,1,2$
- <0 for $y<0$, $y\in(1,2)$
- >0 for $y\in(0,1)$, $y>2$

$y=0$, $y=2$ are unstable, $y=1$ is a stable equilibrium point.



(c) $\frac{dy}{dx} = e^y - 1$

- 0 for $y=0$
- <0 for $y<0$
- >0 for $y>0$

$y=0$ is an unstable equilibrium point