Homework 8 Solutions

1.
$$\int_0^1 \ln x \, dx = \underbrace{x \ln x \Big|_0^1}_{=0 \ (*)} - \underbrace{\int_0^1 x \frac{1}{x}\, dx}_{=1} = -1$$

(*) We interpret this expression as

$$\lim_{r \searrow 0} x \ln x \Big|_r^1 = 1 \cdot \ln 1 - \lim_{r \searrow 0} r \ln r = 0 - 0 = 0 .$$

2. Note first that $\dfrac{1}{\sqrt{1+x^8}}$ does not have vertical asymptotes, so it suffices to check that

$$\int_1^\infty \frac{1}{\sqrt{1+x^8}}\, dx$$

converges.

Now, $$\int_1^r \frac{dx}{\sqrt{1+x^8}} \leq \int_1^r \frac{dx}{x^4} \leq \int_1^\infty \frac{dx}{x^4} = -\frac{1}{3} x^{-3} \Big|_1^\infty = \frac{1}{3}$$

Thus, $\displaystyle\int_1^r \frac{dx}{\sqrt{1+x^8}}$ is a bounded increasing function of r,

which must have a limit as $r \to \infty$.

3. (a) Recall the general "slice method" formula for volume computations,

$$V = \int^b A(x)\, dx$$

a

For the volume of revolution described in the question, the slice is a circle of radius $f(x)$, hence

$$A(x) = \pi f^2(x)$$

$$\Rightarrow V = \pi \int_a^b f^2(x)\, dx$$

(b) $V = \pi \int_0^1 (x^3)^2\, dx = \pi \frac{1}{7} x^7 \Big|_0^1 = \frac{\pi}{7}$

4. (a) $V = \pi \int_1^\infty \left(x^{-\frac{2}{3}}\right)^2 dx = \pi\,(-3)\, x^{-\frac{1}{3}} \Big|_1^\infty = 3\pi$

(b) $A = 2 \int_1^\infty x^{-\frac{2}{3}}\, dx = \pi \cdot 3\, x^{\frac{1}{3}} \Big|_1^\infty = \infty$

So we have a solid with finite volume but infinite cross-sectional area.

5. $W = \int_0^l F(x)\, dx = k \int_0^l x\, dx = \frac{1}{2} k l^2$

Note: we take a positive force because the force that needs to be applied to extend the spring is opposite to the force that the spring exits.