Homework 7 Solutions

1(a). $$\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\, dx$$

$$u = \frac{\pi}{x} \Rightarrow du = -\frac{\pi}{x^2}\, dx$$

$$\Rightarrow \frac{dx}{x^2} = -\frac{du}{\pi}$$

$$= -\frac{1}{\pi} \int \cos u \, du$$

$$= -\frac{1}{\pi} \sin u + C$$

$$= -\frac{1}{\pi} \sin\left(\frac{\pi}{x}\right) + C$$

(b) $$\int_1^e \frac{\ln x}{x}\, dx$$

$$u = \ln x \Rightarrow du = \frac{dx}{x}$$

$$= \int_0^1 u \, du = \frac{1}{2} u^2 \Big|_0^1 = \frac{1}{2}$$

2(a). $$\int \sin x \ln(\cos x)\, dx = -\cos x \ln(\cos x) - \int (-\cos x) \frac{-\sin x}{\cos x}\, dx$$

$$= -\cos x \ln(\cos x) - \int \sin x \, dx$$

$$= \cos x \left(1 - \ln(\cos x)\right) + C$$

(b) Note that

$$\int \sin x \cos x \, dx$$

$$u = \cos x$$
$$du = -\sin x \, dx$$

$$= -\int u \, du = -\frac{1}{2} u^2 + C$$

$$= -\frac{1}{2} \cos^2 x$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} x \sin x \cos x \, dx = \underbrace{-\frac{1}{2} x \cos^2 x \Big|_0^{\frac{\pi}{2}}} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \underbrace{\cos^2 x}_{=\frac{1}{2}(1+\cos 2x)} \, dx$$

$= \frac{1}{2}(1+\cos 2x)$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} dx + \overbrace{\underbrace{\frac{1}{4}\int_0^{\frac{\pi}{2}} \cos(2x)\, dx}_{= \frac{1}{2}\int_0^{\pi} \cos u \, du \;=\; \frac{1}{2} \sin u \Big|_0^{\pi} \;=\; 0}}^{=0}$$

$u = 2x$

$du = 2\,dx$

$$= \frac{\pi}{8}$$

3(a). Note that $\int \sin x \cos^{n-2} x \, dx$ $\qquad u = \cos x \Rightarrow du = -\sin x \, dx$

$$= -\int u^{n-2}\, du = -\frac{1}{n-1} u^{n-1} + C = -\frac{1}{n-1}\cos^{n-1} x + C$$

$$\Rightarrow I_n = \int \cos^n x \, dx = \int (1-\sin^2 x)\cos^{n-2} x \, dx$$

$$= \int \cos^{n-2} x \, dx - \underbrace{\int \sin x \, \sin x \cos^{n-2} x \, dx}_{= \sin x \frac{1}{n-1}\cos^{n-1} x - \int \cos x \frac{1}{n-1} \cos^{n-1} x \, dx}$$

$$\Rightarrow I_n = I_{n-2} + \frac{1}{n-1}\sin x \cos^{n-1} x - \frac{1}{n-1} I_n$$

Now solve for I_n:

$$\underbrace{\left(1 + \frac{1}{n-1}\right)}_{= \frac{n}{n-1}} I_n = I_{n-2} + \frac{1}{n-1}\sin x \cos^{n-1} x$$

$$\Rightarrow I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$$

(b) $I_2 = \frac{1}{2} \sin x \cos x + \frac{1}{2} x + C$

(c) $$\int_0^{2\pi} \cos^4 x \, dx = \underbrace{\frac{1}{4} \sin x \cos^3 x \Big|_0^{2\pi}}_{=0} + \frac{3}{4} \underbrace{\int_0^{2\pi} \cos^2 x \, dx}_{\substack{= \frac{1}{2} \sin x \cos x + \frac{1}{2} x \Big|_0^{2\pi} \\ = \pi}}$$

$$= \frac{3}{4} \pi$$

4. Points of intersection:

$$y = -(1-y^2) - 1 = y^2 - 2 \quad \Rightarrow \quad y^2 - y - 2 = 0$$

$$\Rightarrow y_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1}{2} \pm \frac{3}{2} \qquad \Rightarrow y_1 = -1, \; y_2 = 2$$

Best use y as independent variable, so

$$A = \int_{-1}^{2} (1-y^2) - (-y-1) \, dy = \int_{-1}^{2} (2 + y - y^2) \, dy$$

$$= 2y + \frac{1}{2} y^2 - \frac{1}{3} y^3 \Big|_{-1}^{2} = 2 \cdot 2 + \frac{1}{2} \cdot 4 - \frac{1}{3} \cdot 8 - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= 4 + 2 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3} = \frac{9}{2}$$

5. $$\int_{-a}^{a} f(x) \, dx = \underbrace{\int_{-a}^{0} f(x) \, dx}_{= \int_a^0 f(-u)(-du) = \int_a^0 f(u)\, du = -\int_0^a f(x)\, dx} + \int_0^a f(x) \, dx$$

$u = -x$

$\Rightarrow du = -dx$

$$= \int^{0} f(-u)(-du) = \int^{0} f(u) \, du = -\int^{a} f(x) \, dx$$

$\bar{a}$ a 0

$= 0$