Homework 3 Solutions

1(a) $$\lim_{\theta\to 0}\frac{\theta}{\sin\theta}=\lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}=\frac{1}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}=\frac{1}{1}=1$$

(quotient limit law – applicable here because the limit in the denominator is different from zero.)

(b) $$\lim_{\phi\to 0}\frac{1-2\cos^2\phi}{\phi}$$ does not exist as $1-2\cos^2\phi\to -1$ as $\phi\to 0$, but the denominator goes to zero.

[Note: There was a typo in the question. The intended limit was

$$\lim_{\phi\to 0}\frac{1-\cos^2\phi}{\phi^2}=\lim_{\phi\to 0}\frac{\sin^2\phi}{\phi^2}=\left(\lim_{\phi\to 0}\frac{\sin\phi}{\phi}\right)^2=1$$]

2. $$\exp(x)=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=\lim_{y\to 0}\left(1+y\right)^{\frac{x}{y}}$$

setting $y=\frac{x}{n}$ so that $n=\frac{x}{y}$

$$=\lim_{y\to 0}\left((1+y)^{\frac{1}{y}}\right)^x=\Big(\underbrace{\lim_{y\to 0}(1+y)^{\frac{1}{y}}}_{=:e}\Big)^x$$

(Recall from class that

$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$= \lim_{y \to 0} (1+y)^{\frac{1}{y}}$

Since $\exp(x) = e^x$, we have

$$\exp(a+b) = e^{a+b} = e^a e^b = \exp(a)\exp(b)$$

characteristic relation for the general power function

3. Consider $f(x) = x - \cos x$, which is continuous being the sum of continuous functions.

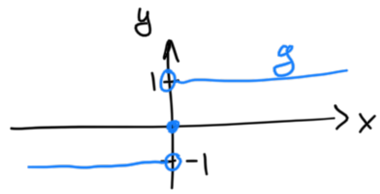
$$f(0) = -1 < 0$$

$$f\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2} - \cos\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2} > 0$$

By the intermediate value theorem, f takes the value 0 for some $x \in (0, \frac{\pi}{2})$, i.e. $x = \cos x$.

4. There are many examples, here a few:

(i) $g(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$



$f(y) = 0$ for all $y \in \mathbb{R}$

$\Rightarrow h(x) = f(g(x)) = 0$ which is obviously continuous,

but g is not continuous

(ii) $g(x) = \begin{cases} x & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$

$$f(y) = \max\{y, 0\} = \begin{cases} y & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

$\Rightarrow h(x) = f(g(x)) = \max\{x, 0\}$ which is continuous.

(iii) $f(y) = e^{-y}$

$g(x) = \frac{1}{x^2}$

g cannot be extended to a continuous function on $\mathbb{R}$,

but

$$h(x) = f(g(x)) = e^{-\frac{1}{x^2}}$$

can be continuously extended to all of $\mathbb{R}$ by defining $h(0) = 0$.

(This example does not quite fit the question as stated, but is a useful example in the wider context.)

5 (a)

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x$$

(b) $$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$h \to 0 \quad h(\sqrt{x+h}+\sqrt{x}) \qquad h \to 0 \quad \sqrt{x+h}+\sqrt{x}$

$$= \frac{1}{2\sqrt{x}} \quad \text{provided } x > 0 .$$

<u>Note</u>: the limit does not exist when $x = 0$, so $f(x)$ is not differentiable at $x=0$ even though it is defined and continuous from the right at this point.