Homework 10 solutions

1(a) Let $w = a \times (b \times c)$

Then (i) $w \perp a \Rightarrow w \cdot a = 0$

(ii) $w \perp b \times c \Rightarrow w = \lambda b + \mu c$ for some $\lambda, \mu \in \mathbb{R}$

Combining (i) and (ii), we see that $\lambda\, b \cdot a + \mu\, c \cdot a = 0$, so

$$\frac{\lambda}{\mu} = -\frac{c \cdot a}{b \cdot a}$$

Moreover, w must be linear in each of a, b, and c (scaling each of the vectors must scale the triple product by the same factor). So the only possibility is $\lambda = c \cdot a$ and $\mu = -b \cdot a$, or the opposite sign. Testing, e.g. taking $a = b = e_1$ and $b = e_2$, using the right-hand rule, confirms the choice of sign as stated.

<u>Note:</u> This solution is elegant, but hard to find. You can always check such identities by computing both sides and comparing.

(b) Use (a):

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b)$$

$$= b\,(a \cdot c) - \cancel{c\,(a \cdot b)} + \cancel{c\,(b \cdot a)} - \cancel{a\,(b \cdot c)} + \cancel{a\,(c \cdot b)} - b\,(c \cdot a)$$

$$= 0$$

2(a) We use the identity $u \cdot (v \times w) = v \cdot (w \times u)$ $\left(= w \cdot (u \times v) \right)$

$$\underbrace{(a \times b)}_{u} \cdot (\underbrace{c}_{v} \times \underbrace{d}_{w}) = \underbrace{c}_{v} \cdot (\underbrace{d}_{w} \times \underbrace{(a \times b)}_{u})$$

$$= c \cdot \big(a\,(d \cdot b) - b\,(d \cdot a) \big)$$

$$= (a \cdot c)(d \cdot b) - (b \cdot c)(a \cdot d)$$

(b) Solution 1: Use 1(c) with $c = a$, $d = b$:

$$\| a \times b \|^2 = (a \cdot a)(b \cdot b) - (b \cdot a)(a \cdot b) = \|a\|^2 \|b\|^2 - (a \cdot b)^2$$

Solution 2: $\| a \times b \|^2 = \|a\|^2 \|b\|^2 \sin^2\theta$, θ is angle between a and b

$$(a \cdot b)^2 = \|a\|^2 \|b\|^2 \cos^2\theta$$

so identity follows from $\sin^2\theta = 1 - \cos^2\theta$.

3. We need one more direction vector for the plane, e.g.

$$w = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$$

So a possible parametric representation of the plane is

$$x = \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$$

(Others are possible!)

4. Line: $x = a + \lambda v$

Square distance to p: $f(\lambda) = \| p - (a + \lambda v) \|^2$

$$= \| (p - a) - \lambda v \|^2$$

$$= \| p - a \|^2 - 2\lambda\, v \cdot (p - a) + \lambda^2 \|v\|^2$$

To find the minimum, we differentiate :

$$f'(\lambda) = -2\, v \cdot (p-a) + 2\lambda \|v\|^2 = 0$$

$$\Rightarrow \lambda = \frac{v \cdot (p-a)}{\|v\|^2}$$

$$v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \|v\|^2 = 2$$

$$p - a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$v \cdot (p-a) = 2 \cdot 1 - 1 \cdot 1 + 0 = 1$$

$$\Rightarrow f_{min} = \Big\| p - a - \underbrace{\frac{v \cdot (p-a)}{\|v\|^2}\, v}_{\circledast} \Big\|^2$$

$$= \| (2,1,-3) - \tfrac{1}{2}(1,-1,0) \|^2$$

$$= \| (\tfrac{3}{2}, \tfrac{3}{2}, -3) \|^2$$

$$= \frac{9}{4} + \frac{9}{4} + 9 = \frac{54}{4}$$

$$\Rightarrow d = \sqrt{\frac{54}{4}} = \frac{\sqrt{54}}{2}$$

$\circledast$ This formula is the projection of $p-a$ on the direction $\hat{v}$, which is still to be discussed in class.

5. Suppose

$$W = \sum_{k=1}^{n} \alpha_k v_k = \sum_{k=1}^{n} \beta_k v_k$$

$$\Rightarrow \sum_{k=1}^{n} (\alpha_k - \beta_k)\, v_k = 0$$

Since $\{v_k\}$ l.i., this implies $\alpha_k - \beta_k = 0 \Rightarrow \alpha_k = \beta_k$.