Calculus and Elements of Linear Algebra I

Final Exam

December 14, 2020

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Some trigonometric identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin 2x = 2\,\sin x\,\cos x$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \cos 2x = 2\,\cos^2 x - 1 = 1 - 2\,\sin^2 x$$

$$1 + \cot^2\theta = \csc^2\theta \qquad \tan 2x = \frac{2\,\tan x}{1 - \tan^2 x}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Useful integrals:

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$$

$$\int \frac{du}{1+u^2} = \arctan u + C = -\operatorname{arccot} u + C'$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}|u| + C$$

$$\int \sec u\,du = \ln\left|\sec u + \tan u\right| + C$$

1. Compute the limits

(a) $\lim_{t\to\infty} e^{-t}\cos(t)\sin(t)$

(b) $\lim_{\theta\to 0} \dfrac{1-\cos(2\theta)}{\theta^2}$

(c) Compute the derivative of

$$f(x) = \frac{1}{x}$$

directly from its definition as the limit of a difference quotient.

(5+5+5)

(a) $\left| e^{-t} \cos t \sin t \right| = |e^{-t}| \underbrace{|\cos t|}_{\leq 1} \underbrace{|\sin t|}_{\leq 1} \leq e^{-t} \to 0$ as $t\to\infty$.

hence, by the squeeze law, $\lim_{t\to\infty} e^{-t} \cos t \sin t = 0$.

(b) $\lim_{\theta\to 0} \dfrac{1-\cos 2\theta}{\theta^2} = \lim_{\theta\to 0} \dfrac{2\sin^2\theta}{\theta^2} = 2$

where we use the standard trigonometric limit $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

(c) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$= \lim_{h\to 0} \dfrac{x-(x+h)}{h\,x\,(x+h)} = \lim_{h\to 0} \dfrac{-1}{x(x+h)} = -\dfrac{1}{x^2}$

2. Consider the function

$$f(x) = \frac{2-x^2}{1-x^2}.$$

What is the domain of f? Find horizontal and vertical asymptotes, local minima, local maxima, and inflection points of f. Identify the regions where the graph of f is concave upward or concave downward. Finally, sketch the graph into the coordinate system provided. (15)

- $D(f) = \mathbb{R} \setminus \{\pm 1\}$

- $\lim_{x \to \pm\infty} f(x) = 1$, so hor. asymptote is $y = 1$

- vertical asymptotes at $x = \pm 1$, note that $\lim_{|x| \searrow 1} f(x) = -\infty$,

  $\lim_{|x| \nearrow 1} f(x) = +\infty$.

- $f'(x) = \dfrac{-2x(1-x^2) - (-2x)(2-x^2)}{(1-x^2)^2} = \dfrac{2x}{(1-x^2)^2}$

critical point: $x = 0$

For $x > 0$, $f'(x) > 0 \Rightarrow f$ is increasing

$x < 0$, $f'(x) < 0 \Rightarrow f$ is decreasing

$\Big\} \Rightarrow$ loc. min at $x = 0$

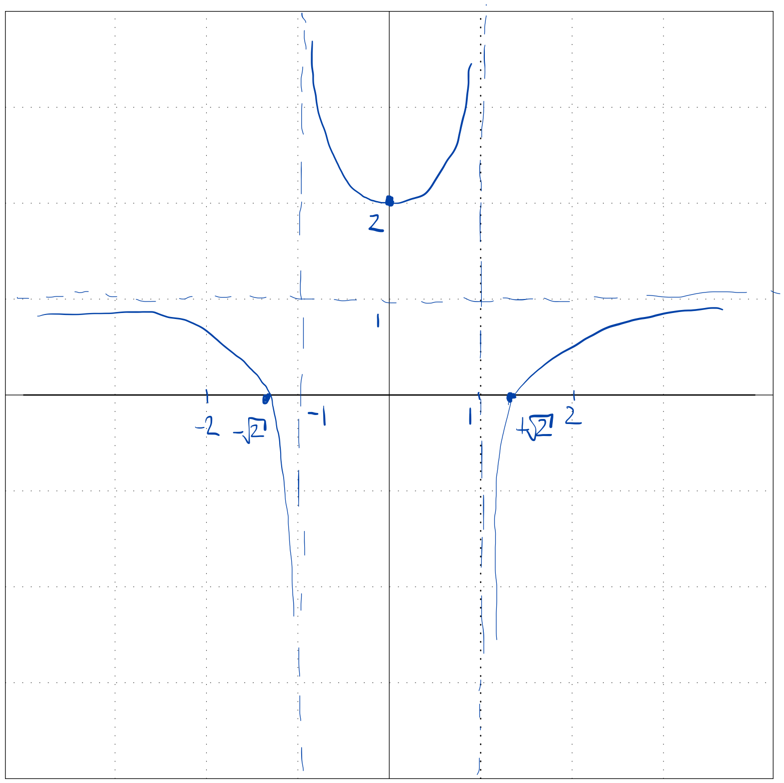
- $f''(x) = \dfrac{2(1-x^2)^2 - (-2x)2(1-x^2)2x}{(1-x^2)^4} = 2\dfrac{1-x^2+4x^2}{(1-x^2)^3}$

$= 2\dfrac{1+3x^2}{(1-x^2)^3}$

For $|x| < 1$, $f''(x) > 0 \Rightarrow f$ concave up

$|x| > 1$, $f''(x) < 0 \Rightarrow f$ concave down

$\Rightarrow$ no point of inflection.



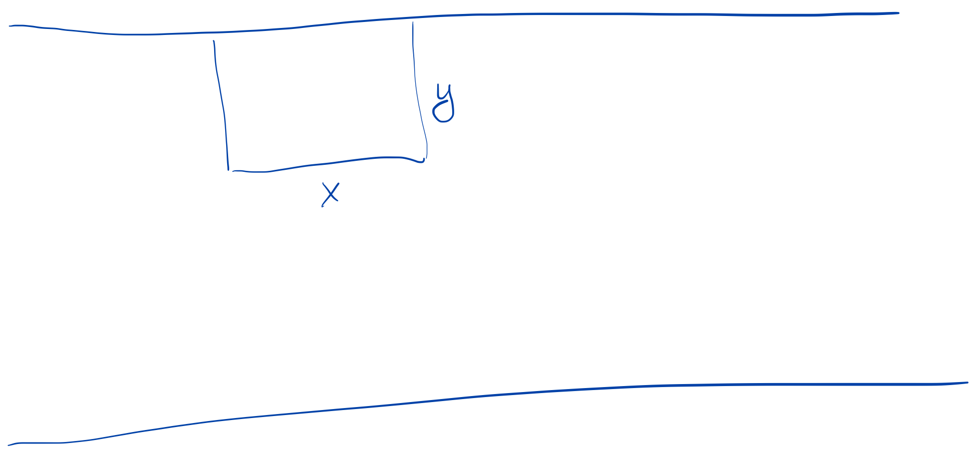
3. A farmer owns a 10 km long stretch of land between two parallel rivers that are 2 km apart. What is the area of the largest rectangular corral he can enclose with

   (a) 2 km of fencing,

   (b) 5 km of fencing,

   assuming that no fence is needed along the river? (7+3)

(a)



$x + 2y = L = 2\,km \quad \Rightarrow \quad y = \dfrac{L-x}{2}$

$A = xy = \frac{1}{2} x (L - x)$

$\Rightarrow \dfrac{dA}{dx} = -x + \frac{1}{2}L = 0 \quad \Rightarrow \quad x = \frac{1}{2}L \quad \Rightarrow \quad y = \dfrac{L - \frac{1}{2}L}{2} = \frac{1}{4}L$

$\Rightarrow$ The corral has a length of $x = 1\,km$ along one of the rivers and a width of $y = 0.5\,km$.

(b) With only 4 km of fencing, he can enclose the entire stretch of land he owns.

4. Use implicit differentiation to find an equation for the tangent line to the graph of $\sin(x+y) = y^2\cos(x)$ at point $(0,0)$. (5)

$$\left(1+\frac{dy}{dx}\right)\cos(x+y) = 2y\frac{dy}{dx}\cos x + y^2(-\sin x)$$

At $x=0$, $y=0$:

$$\left(1+\frac{dy}{dx}\right)\underbrace{\cos 0}_{=1} = 0 + 0 \quad \Rightarrow \quad \frac{dy}{dx} = -1$$

$\Rightarrow$ The tangent line equation is $y=-x$

5. Compute the following integrals:

(a) $\int_0^1 \ln x \, dx$

(b) $\int \frac{x^2+1}{x^2-1} \, dx$

(5+10)

(a) $\int_0^1 \ln x \, dx = \underbrace{x \ln x \Big|_0^1}_{=0} - \underbrace{\int_0^1 x \frac{1}{x} \, dx}_{=1} = -1$

(b) $(x^2+1):(x^2-1) = 1 + \frac{2}{x^2-1}$

$\underline{-|x^2-1}$

2

Partial fractions: $\frac{2}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{x^2-1}$

$\Rightarrow A+B=0$

$-A+B=2$

$\Rightarrow A=-1, B=1$

$\Rightarrow \int \frac{x^2+1}{x^2-1} \, dx = \int \left(1 - \frac{1}{x+1} + \frac{1}{x-1}\right) dx = x - \ln(x+1) + \ln(x-1) + C$

$= x + \ln \frac{x-1}{x+1} + C$

6. Is the following improper integral convergent? There is no need to compute the answer, but you should give detailed reasoning.

$$\int_0^\infty \frac{\ln x + e^{-x}}{1+x^2}\, dx$$

(10)

The integrand has a vertical asymptote at $x=0$. So let's check $x \searrow 0$ and $x \to \infty$ independently:

$$I_1 = \int_0^1 \frac{\ln x + e^{-x}}{1+x^2}\, dx = \int_0^1 \frac{\ln x}{1+x^2}\, dx + \underbrace{\int_0^1 \frac{e^{-x}}{1+x^2}\, dx}_{\text{cont. on } [0,1],\text{ so no issue here}}$$

Now
$$\int_0^1 \frac{\ln x}{1+x^2}\, dx > \int_0^1 \ln x\, dx = -1 \quad \text{by Q5(a).}$$

As $\frac{\ln x}{1+x^2}$ does not change sign on $[0,1]$, I_1 converges.

$$I_2 = \int_e^\infty \frac{\ln x + e^{-x}}{1+x^2}\, dx \leq 2\int_e^\infty \frac{\ln x}{x^2}\, dx$$

which converges since $\int_e^\infty \frac{1}{x^\alpha}\, dx$ converges for $\alpha > 1$.

As the integrand does not change sign on $[1,\infty)$, I_2 also converges.

Note: the upper bound can also be computed explicitly (not required):

$$\int_e^\infty \frac{\ln x}{x^2} = \underbrace{-\frac{1}{x}\ln x \Big|_e^\infty}_{=\frac{1}{e}} + \underbrace{\int_e^\infty x^{-3}\, dx}_{=-\frac{1}{2}x^{-2}\Big|_e^\infty = \frac{1}{2}e^2} = \frac{1}{e} + \frac{1}{2}e^2$$

7. Consider the differential equation

$$\frac{dy}{dt} = t^3 y^3 .$$

(a) Solve the initial value problem with $y(0) = 2$.

(b) Does this equation have equilibrium points? Are they stable or unstable?

(10+5)

(a) $$\int_2^{y(t)} \frac{dy}{y^3} = \int_0^t t^3\, dt$$

$$\Rightarrow \quad -\frac{1}{2} y^{-2} \Big|_2^{y(t)} = \frac{1}{4} t^4 \Big|_0^t$$

$$\Rightarrow \quad \frac{1}{8} - \frac{1}{2} \frac{1}{y^2(t)} = \frac{1}{4} t^4$$

$$\Rightarrow \quad \frac{1}{4} - \frac{1}{2} t^4 = \frac{1}{y^2(t)}$$

$$\Rightarrow \quad y(t) = \frac{1}{\sqrt{\frac{1}{4} - \frac{1}{2} t^4}}$$ (choose pos. root to match initial cond.!)

(b) For $t>0$, $\frac{dy}{dt} = 0$ iff $y=0$,

$\frac{dy}{dt} > 0$ for $y>0$,

$\frac{dy}{dt} < 0$ for $y<0$

$y=0$ is the only equilibrium point, it is unstable.

8. Show that, for $u, v \in \mathbb{R}^3$,

$$||\mathbf{u}||^2 \, ||\boldsymbol{v}||^2 = (\mathbf{u} \cdot \boldsymbol{v})^2 + ||\mathbf{u} \times \boldsymbol{v}||^2 \, .$$

(5)

$u \cdot v = ||u|| \, ||v|| \cos \theta$ $\qquad$ θ : angle between u and v

$||u \times v|| = ||u|| \, ||v|| \sin \theta$

$\Rightarrow (u \cdot v)^2 + ||u \times v||^2 = ||u||^2 \, ||v||^2 (\cos^2\theta + \sin^2\theta) = ||u||^2 \, ||v||^2$

9. Find the general solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 2 & 0 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -2 \\ -2 \\ 0 \\ -3 \end{pmatrix}.$$

(10)

$$\left(\begin{array}{cccc|c} 2 & 0 & 2 & 4 & -2 \\ 0 & 1 & 0 & 1 & -2 \\ 2 & -1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 3 & -3 \end{array}\right) \xrightarrow[\substack{R3-R1 \to R3 \\ R4 - \frac{R1}{2} \to R4}]{\frac{R1}{2} \to R1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 & -2 \end{array}\right)$$

$$\xrightarrow[R2-R4 \to R4]{R2+R3 \to R3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow x = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

Check (not required):

$$A \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+0 \\ 0+-2 \\ -2+2 \\ -1-2 \end{pmatrix} = b \checkmark, \quad A \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2-2 \\ 0-0 \\ 2-2 \\ 1-1 \end{pmatrix} = 0 \checkmark$$

$$A \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4+0-4 \\ 0+1-1 \\ 4-1-3 \\ 2+1-3 \end{pmatrix} = 0 \checkmark$$

10. Let $L: \mathbb{R}^4 \to \mathbb{R}^4$ be the "shift mapping" defined as follows:

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

(a) Show that L is a linear transformation on $\mathbb{R}^4$.

(b) Write out the matrix S which represents L with respect to the standard basis.

(c) Find a basis for Range S and Ker S.

(d) State the "rank-nullity theorem" and verify explicitly that the result obtained in part (c) matches the statement of the theorem.

(5+5+5+5)

(a) $$L(\lambda x+\mu y) = L\begin{pmatrix} \lambda x_1+\mu y_1 \\ \vdots \\ \lambda x_5+\mu x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda x_1+\mu x_1 \\ \vdots \\ \lambda x_4+\mu x_4 \end{pmatrix} = \lambda\begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_4 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_4 \end{pmatrix}$$

$$= \lambda Lx + \mu Ly$$

(b) $$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(c) We immediately see that columns 1–4 have pivots, so

Range S = span $\{e_2, e_3, e_4, e_5\}$

Ker S = span $\{e_5\}$

(d) Rank-nullity theorem for an $n \times m$ matrix S says that

$$\operatorname{rank} S + \operatorname{nullity} S = m$$

or $\dim \operatorname{Range} S + \dim \operatorname{Ker} S = m$.

Here: $4 \quad + \quad 1 \quad = 5$.