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Problem 1 [2 x 8 Points]: Consider the function

$$f(x) = \begin{cases} 4x^2 + 6x & \text{if } x \geq 1, \\ -x + k & \text{if } x < 1. \end{cases}$$

a) Find $k \in \mathbb{R}$ such that f is continuous on whole $\mathbb{R}$. Show that f is continuous on whole $\mathbb{R}$ for the selected value of k.

b) Using the value of k found in a) and using the definition of the derivative as the limit of the difference quotient prove or disprove that f is differentiable in $x = 1$.

Hint: You will ***not*** *get credit for just applying rules for differentiation. You* ***must use the definition of derivative as limit of difference quotient.***

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Problem 2 [3 x 6 Points]: Consider the problem of calculating the area bounded by the curve $y = 1+x$, the x-axis, and the lines $x = 0$ and $x = 1$:

a) Create a regular partition of $[0, 1]$ into $n \in \mathbb{N}$ sub-intervals, and let x_i^* be the right-hand endpoint of each sub-interval. Write down the subintervals and the points x_i^*.

b) Estimate A_0^1 for $4, 8, 16$ and 32 sub-intervals.

c) Find the limit of A_0^1 for $n \to \infty$.

Hint: $\sum_{i=0}^{n} i = \frac{1}{2} n(n+1)$.

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Problem 3 [2 x 8 Points]: For the following sets of vectors, find the condition on the parameter $b \in \mathbb{R}$ such that the vectors are linearly independent:

a) $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -b \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ b \end{pmatrix} \right\}$

b) $\left\{ 1 + x^2 + bx^3,\ x + x^2,\ 1 + bx - x^3 \right\}$

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Problem 4 [18 Points]: On the real vector space of polynomials of degree 2 over the interval $[-1, 1]$, consider the following inner product and norm

$$< u, v > := \int_{-1}^{1} u(x)v(x)\, dx \ , \qquad \|u\| := \sqrt{< u, u >}^2.$$

In this space, are the vectors $\{1, x, \frac{1}{2}(3x^2 - 1)\}$ orthonormal? In case they are not orthonormal, how do they need to be scaled to become orthonormal? Give reasoning for all parts of your answer.

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Problem 5 [2 x 8 Points]: Consider

$$A = \begin{pmatrix} 1 & 4 & 3 & 2 & 5 \\ 4 & 8 & 12 & 9 & 0 \\ 3 & 4 & 9 & 7 & -5 \\ 2 & 8 & 6 & 5 & 6 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 2 \\ 8 \\ 6 \\ 4 \end{pmatrix}.$$

a) Determine the rank, nullspace, and nullity of the matrix A.

b) Find a particular solution $\vec{p}$ for the non-homogeneous system $A\vec{x} = \vec{b}$. Use $\vec{p}$ and your result from a) to describe the general solution to this non-homogeneous system.

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Problem 6 [2 x 8 Points]: Compute the following integrals:

a) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$

b) $\displaystyle\int_0^3 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

Problem 7 [6 + 2 + 2 Bonus Points]:

a) Prove the reduction formula

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

Hint: You will need integration by parts *and the fact that* $\cos^2(x) + \sin^2(x) = 1$.

b) Use part a) to evaluate $\int \cos^2(x)\,dx$.

c) Use parts a) and b) to evaluate $\int \cos^4(x)\,dx$.

5(a)

$$\underbrace{\left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 5 & 2 \\ 4 & 8 & 12 & 9 & 0 & 8 \\ 3 & 4 & 9 & 7 & -1 & 6 \\ 2 & 8 & 6 & 5 & 6 & 4 \end{array}\right)}_{A} \xrightarrow[\substack{-3R1+R3 \to R3 \\ -2R1+R4 \to R4}]{-4R1+R2 \to R2} \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 5 & 2 \\ 0 & -8 & 0 & 1 & -20 & 0 \\ 0 & -8 & 0 & 1 & -16 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{-R4+R2 \to R2 \\ -R4+R3 \to R3}]{-2R4+R1 \to R1} \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 0 & 13 & 2 \\ 0 & -8 & 0 & 0 & -16 & 0 \\ 0 & -8 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \end{array}\right) \xrightarrow[\frac{R2}{-8} \to R2]{R3-R2 \to R3} \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 0 & 13 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{-4R2+R1 \to R1} \left(\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

rank $A = 4$

$\operatorname{Ker} A = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$

nullity $A = 1$

gen. solution: $x = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

check: $A \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \ldots = 0$ (check for yourself...)

- **Write your answers in the booklet, structure your solutions well!**
- **Give reasoning and clearly indicate your final answer or conclusion!**
- **Blue or black pen, no pencil!**
- **No calculator with functions for differentiation, integration, vectors, matrices!**

Problem 1 [8 + 10 Points]: Given the following functions $f(x)$, find $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

a) $f(x) = x^2 - 4$,

b) $f(x) = 4x^3 + 3x^2 + x$.

*Hint: You will **not** get credit for applying rules for differentiation. You **must calculate the limit of the difference quotient.***

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Problem 2 [12 Points]: Compute the following integral, using ~~integration by parts and substitution.~~

$$\int x^7 \sqrt{5 + 3x^4}\, dx.$$

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Problem 3 [6 + 10 + 8 Points]: Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$, where $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_2 - x_3 \\ 2x_2 + 4x_3 \\ x_1 + x_3 \end{pmatrix}$.

a) Find the standard matrix A (i.e. for the Euclidean bases) associated with T.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \\ 1 & 0 & 1 \end{pmatrix}$$

b) Determine nullity and rank of A. Give reasoning.

c) Determine the nullspace of A. Give reasoning. since nullity $A = 0$, Ker $A = \{0\}$

(b): $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & -2 \end{pmatrix} \xrightarrow[R4-R2]{R3-R2 \to R3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R4 \to R3]{R4-3R3 \to R4} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$ rank $A = 3$, nullity $A = 0$ as rank A + nullity $A = 3$ (# columns of A)

Problem 4 [2 + 4 + 6 Points]: Consider the vector space V of functions $f : [0,1] \to \mathbb{R}$ which are continuously differentiable. We define an inner product by

$$\langle f, g \rangle := \int_0^1 \Big(f(x)g(x) + f'(x)g'(x) \Big)\, dx \qquad \text{for } f, g \in V.$$

Show that for any $f, g, h \in V$ and any $\lambda, \mu \in \mathbb{R}$

a) $\langle f, g \rangle = \langle g, f \rangle$,

b) $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$.

c) In this inner product, are $f(x) = \sin(\pi x)$ and $g(x) = x - \frac{1}{2}$ orthogonal? Prove or disprove.

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Problem 5 [2 x 8 Points]: Are the following sets of vectors linearly dependent? Prove or disprove.

a) $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 5 \end{pmatrix} \right\}$

b) $\left\{ 1 + t - t^3, -2 + 3t - t^2 + 2t^3, 1 + t^2 + 5t^3 \right\}$

(a) $\begin{pmatrix} 1 & -2 & 1 \\ -1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 4 \end{pmatrix}$ $\Rightarrow$ rank $A = 2 < 3$
$\Rightarrow$ columns are l.d.

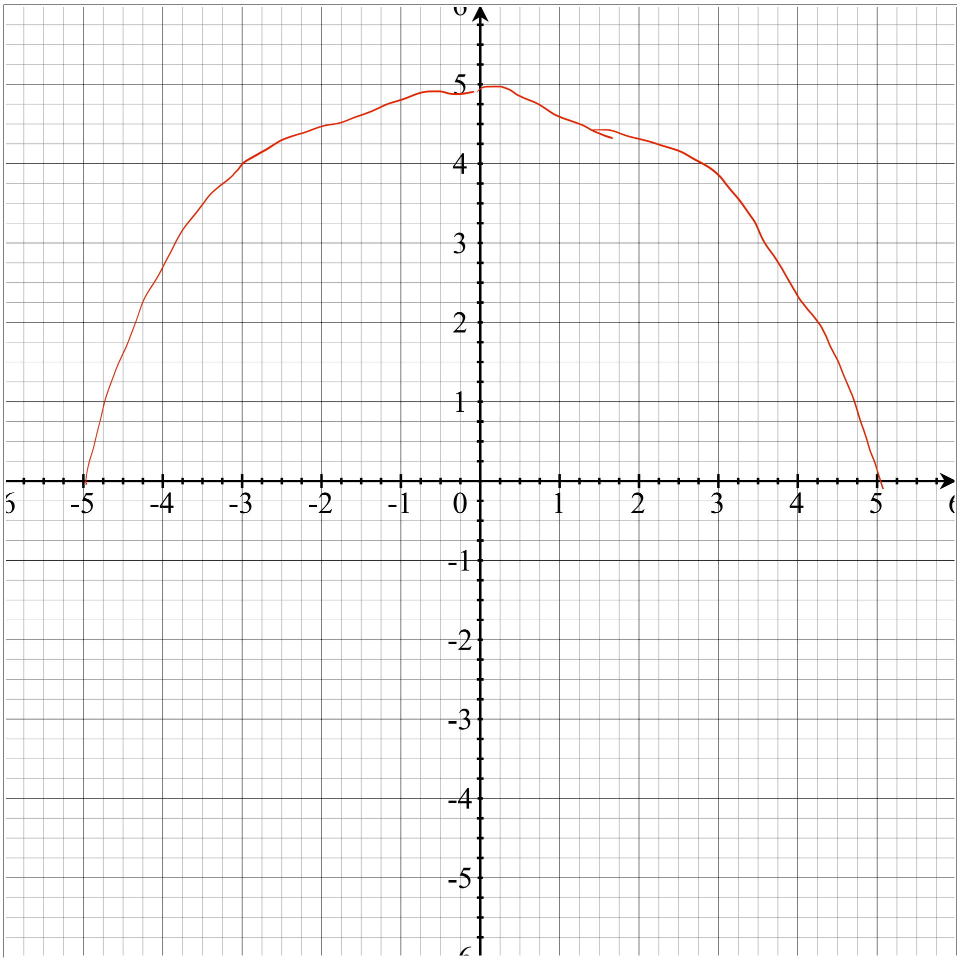
(b) $\begin{pmatrix} 1 & -2 & 1 \\ 1 & 3 & 0 \\ 0 & -1 & 1 \\ -1 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 6 \end{pmatrix}$ rank $A = 3$, vectors are l.i.

$y^2 = 25 - x^2 \Rightarrow x^2 + y^2 = 25$

Problem 6 [8 + 10 Points]: Given $f(x) = \sqrt{25 - x^2}$ consider the Mean Value Theorem (MVT) for this function over the interval $[-3, 5]$.

a) Sketch the graph of this function in the coordinate system below. Label the points that determine the secant relevant to the application of the Mean Value Theorem.

b) Find the value $c \in [-3, 5]$ which is guaranteed to exist by the theorem. Place the point $(c, f(c))$ in the coordinate system below.



Problem 7 [2+2+6 Bonus Points]: For $n \in \mathbb{N}$ consider the identity

$$\int_0^{\pi} \sin^{2n}(x)\, dx = \frac{(2n)!}{(n!)^2} \frac{\pi}{2^{2n}}.$$

a) Prove the identity for the case $n = 1$.

b) Using a), prove the identity for $n = 2$.

c) Assuming that the identity is true for some fixed $n \in \mathbb{N}$, prove that it is also true for $n + 1$.

Hint: You will need integration by parts *and the fact that* $\cos^2(x) + \sin^2(x) = 1$.