Matrix inverse

$A \in M(n \times n)$

- $AA^{-1} = I$
- $A^{-1}A = I$

$$A \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix}$$

- A^{-1} is unique <u>when it exists</u>

  $\iff$ rank $A = n$ $\iff$ every row (column) has a pivot

  $\iff$ $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$

  $\Rightarrow$ if $B \in M(n \times n)$ with the property that $AB = I$, then $B = A^{-1}$

- $(A^{-1})^{-1} = A$

  check: $\underbrace{(A^{-1})^{-1}}_{B} A^{-1} = I$ because $(A^{-1})^{-1}$ is the inverse of A^{-1}

  $BA^{-1} = I \Rightarrow B = A$ by unique solvability and $AA^{-1} = I$

- $(A^{-1})^T = (A^T)^{-1}$

  check: $\underbrace{(A^{-1})^T}_{B} A^T = (AA^{-1})^T = I^T = I \Rightarrow B = (A^T)^{-1}$

- $(AB)^{-1} = B^{-1}A^{-1}$

  check: $\underbrace{B^{-1}A^{-1}}_{C} AB = B^{-1} I B = B^{-1}B = I \Rightarrow C = (AB)^{-1}$

Application: Change & Basis

$V = \text{span}\{1, x, x^2\}$ "vector space of polynomials of degree ≤ 2"

$1 = e_1$, $x = e_2$, $x^2 = e_3$

Take second basis $b_1 = 2x - 1$, $b_2 = 2x + 1 + x^2$, $b_3 = x^2 - 1$

The new basis has coordinates with respect to e_1, e_2, e_3: $\underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{v_3}$

$\begin{pmatrix} x_1 \\ \vdots \end{pmatrix}$ ... its coordinates u w.r.t. b_1, b_2, b_3?

If $p(x)$ has coordinates $x = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$ w.r.t. e_1, e_2, e_3, what are the coordinates ...

$$x = y_1 b_1 + y_2 b_2 + y_3 b_3 \quad \text{or} \quad Sy = x \qquad S = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$$

$$\Rightarrow y = S^{-1} x$$

Here, to compute S^{-1}:

$$\left(\underbrace{\begin{matrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{matrix}}_{S} \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right.\right) \xrightarrow[2R1+R2 \to R2]{-R1 \to R1} \left(\begin{matrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 1 & 1 \end{matrix} \left| \begin{matrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right.\right)$$

$$\xrightarrow[\substack{R3 \to R2 \\ -4R3+R2 \to R3}]{R3+R1 \to R1} \left(\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -6 \end{matrix} \left| \begin{matrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & -4 \end{matrix}\right.\right) \xrightarrow[\frac{R3}{-6} \to R3]{} \left(\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \left| \begin{matrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{matrix}\right.\right)$$

$$\xrightarrow[-2R3+R1 \to R1]{-R3+R2 \to R2} \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \left| \underbrace{\begin{matrix} -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{matrix}}_{S^{-1}}\right.\right)$$

Let $p(x) = 2x^2 - x + 4$ with coordinates w.r.t. e_1, e_2, e_3: $x = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

$p'(x) = 4x - 1$

Coordinates w.r.t. basis b_1, b_2, b_3: $y = S^{-1}x = \begin{pmatrix} -\frac{4}{3} - \frac{1}{3} - \frac{2}{3} \\ \frac{4}{3} - \frac{1}{6} + \frac{2}{3} \\ -\frac{4}{3} + \frac{1}{6} + \frac{4}{3} \end{pmatrix} = \begin{pmatrix} -\frac{7}{3} \\ \frac{11}{6} \\ \frac{1}{6} \end{pmatrix}$

So: $p(x) = -\frac{7}{3}(2x-1) + \frac{11}{6}(2x+1+x^2) + \frac{1}{6}(x^2-1) = 2x^2 + \dots$

<u>Recall:</u> Differentiation is represented w.r.t. e_1, e_2, e_3 by

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$e_1' = 0 \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$e_2' = 1 \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$e_3' = 2x \quad \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

<u>Q:</u> What matrix represents differentiation w.r.t. b_1, b_2, b_3?

$$S^{-1}DS = \underbrace{\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & 1 & \frac{1}{3} \\ -\frac{2}{3} & -1 & -\frac{1}{3} \end{pmatrix}$$

$$S^{-1} D S \begin{pmatrix} -\frac{7}{3} \\ \frac{11}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{14}{9} + \frac{1}{9} \\ -\frac{14}{9} + \frac{11}{6} + \frac{1}{18} \\ \frac{14}{9} - \frac{11}{6} - \frac{1}{18} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{-28+33+1}{18} = \frac{6}{18} = \frac{1}{3} \\ \frac{28-33-1}{18} = -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$p'(x) = \frac{5}{3}(2x-1) + \frac{1}{3}(2x+1+x^2) - \frac{1}{3}(x^2-1)$$

$$= \frac{10}{3}x - \frac{5}{3} + \frac{2}{3}x + \frac{1}{3} + \cancel{\frac{1}{3}x^2} - \cancel{\frac{1}{3}x^2} + \frac{1}{3} = -1 + 4x$$

Outlook next semester:

$$A = S^{-1} \Lambda S$$

diagonal

"Diagonalization"

$$Av = \lambda v$$

eigenvalue

eigenvector

$$\underbrace{(A - \lambda I)}_{\text{needs to be singular}} v = 0$$

Abstract view on change-of-variable (see above):

$$\begin{tikzcd} (V,E) \arrow[r, "D"] & (V,E) \\ (V,B) \arrow[u, "S"] \arrow[r, "T"'] & (V,B) \arrow[u, "S"'] \end{tikzcd}$$

$$T = S^{-1} D S$$