Gaussian elimination revisited:

$$Ax = b \quad \text{with} \quad A = \begin{pmatrix} 2 & -2 & -4 & -1 & 5 \\ 3 & -3 & -6 & 3 & 12 \\ 3 & -3 & -6 & 2 & 11 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 21 \\ 18 \end{pmatrix}$$

augmented matrix

$$\left(\begin{array}{ccccc|c} 2 & -2 & -4 & -1 & 5 & 5 \\ 3 & -3 & -6 & 3 & 12 & 21 \\ 3 & -3 & -6 & 2 & 11 & 18 \end{array}\right) \xrightarrow[R1 \to R2]{\frac{R2}{3} \to R1} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 1 & 4 & 7 \\ 2 & -2 & -4 & -1 & 5 & 5 \\ 3 & -3 & -6 & 2 & 11 & 18 \end{array}\right)$$

$$\xrightarrow[-3R1+R3 \to R3]{-2R1+R2 \to R2} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 1 & 4 & 7 \\ 0 & 0 & 0 & -3 & -3 & -9 \\ 0 & 0 & 0 & -1 & -1 & -3 \end{array}\right) \xrightarrow[\frac{R2}{-3} \to R2,\ \frac{R2}{-3} + R3 \to R3]{R3+R1 \to R1} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pivots

"missing pivots"

Solution is

$$x = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

In general, if not all columns have pivots, expect that a solution exists only for some vectors b and the given matrix A.

Def.:
- The first non-zero entry of a row is called a <u>pivot</u> provided there is no other pivot in that column.
- Let $A \in M(n \times m)$. Then

$$\ker A = \{ v \in \mathbb{R}^m : Av = 0 \}$$

"kernel" or "null space" or solution space of homogeneous equation. (It's a vector space!)

$$\operatorname{Range} A = \{ Ax : x \in \mathbb{R}^m \}$$

rank A: # of pivots after Gaussian elimination

Let $v_1, \dots, v_m \in \mathbb{R}^n$, $b \in \mathbb{R}^n$

Goal: write $b = x_1 v_1 + \dots + x_m v_m$

same as writing $Ax = b$ with $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{pmatrix}$ $A \in M(n \times m)$

Observations:
- columns without a pivot do not contribute to Range A.

  → To select a basis for the column space (= Range A), use Gaussian elimination to find which columns have pivots, then choose original columns as basis vectors.

- Columns without a pivot contribute to a basis for $\underbrace{\operatorname{Ker} A}_{\text{vector space of solutions to } Av=0}$.

- the vectors for the span of Ker A that come from above are l.i. by construction, so are a basis.

$$\boxed{\underbrace{\dim \operatorname{Ker} A}_{\text{nullity } A} + \underbrace{\dim \operatorname{Range} A}_{\text{rank } A} = m}$$

"rank-nullity theorem"

- $Ax = b$ has a solution for every $b \in \mathbb{R}^n$ iff $\dim \operatorname{Range} A = n$
- $Ax = b$ has a unique solution if $b \in \operatorname{Range} A$ and $\dim \operatorname{Ker} A = 0$
- $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$ iff

  $n = m = \operatorname{rank} A$

In that case, we say A is non-singular or invertible and we write

$$\begin{aligned} x &= A^{-1} b \\ &= A^{-1}(b_1 e_1 + b_2 e_2 + \dots + b_n e_n) \\ &= b_1 A^{-1} e_1 + \dots + b_n A^{-1} e_n \end{aligned}$$

$e_i = (0, \dots, 1, 0 \dots)$ with the 1 at position i

This gives rise to the following procedure for computing A^{-1}:

$(A \mid I) \to \dots \to (I \mid A^{-1})$

$(0 \ \tfrac{1}{2} \ -\tfrac{1}{2}$

E.g. $\quad A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-2R2+R3 \to R3]{\substack{R2 \leftrightarrow R1 \\ 2R1 \to R2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{-\frac{1}{2}R2+R3 \to R3 \\ -2R3 \to R3}]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -1 & -2 & 1 \end{array}\right) \begin{array}{c} \\ \\ \\ 1 \quad | \quad 2 \quad 4 \quad -2 \end{array} \xrightarrow[-R3+R1 \to R1]{R3+R2 \to R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 2 \\ 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right)$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \quad \underbrace{\begin{matrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{matrix}}_{A^{-1}}$$

Check: $\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 0 \cdot (-2) + 4 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} & \cdots \\ 1 \cdot (-2) + 0 \cdot 4 + 1 \cdot 2 & \cdots \\ \vdots & \cdots \end{pmatrix} = \begin{pmatrix} 1 & \cdots \\ 0 & \cdots \\ \cdots & \cdots \end{pmatrix}$

$= I$