Matrix multiplication

$A = (a_{ij})$ $\quad i = 1, \dots, n \quad j = 1, \dots, m \qquad n$: # rows, $\; m$: # columns $\qquad A \in M(n \times m)$

$B = (b_{jk})$ $\quad j = 1, \dots, m \quad k = 1, \dots, p \qquad B \in M(m \times p)$

$$(AB)_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}}_{B} = \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 5 \end{pmatrix}$$

In general: $AB \neq BA$

Transpose of a matrix: $(A^T)_{ij} = (a_{ji})$ $\qquad$ e.g. $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 3 & 1 \end{pmatrix}$

In particular: $x \cdot y = x^T y = (x_1, \dots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

For matrices with complex entries: "Hermitian conjugate" $\quad A^H = (A^*)^T$

$$x \cdot y = x^H y$$

E.g. $\begin{pmatrix} i & -i \\ 1 & 2 \end{pmatrix}^H = \begin{pmatrix} -i & 1 \\ i & 2 \end{pmatrix}$

Fact: $\boxed{(AB)^T = B^T A^T}$ $\qquad (AB)^H = B^H A^H$

Systems of linear equations

$$\begin{aligned} x_2 + 2x_3 - x_4 &= 1 \\ x_1 + x_3 + x_4 &= 4 \\ -x_1 + x_2 - x_4 &= 2 \\ 2x_2 + 3x_3 - x_4 &= 7 \end{aligned}$$

write as

$$\underbrace{\begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 4 \\ 2 \\ 7 \end{pmatrix}}_{b}$$

So a system of linear equations is a matrix equation of the form

$$Ax = b$$

"inhomogeneous equation" for $b \neq 0$

with x the vector of unknowns.

Suppose x and y are two solutions, i.e.

$$\begin{array}{rl} & Ax = b, \\ -) & Ay = b \\ \hline & A(x-y) = 0 \end{array}$$

So the difference vector $x - y = v$ between the solution satisfies

$$Av = 0$$

"homogeneous equation"

Vice versa, if v solves $Av = 0$, x solves $Ax = b$, then

$$y := x + \lambda v \quad \text{solves} \quad Ay = A(x + \lambda v) = \underbrace{Ax}_{b} + \lambda \underbrace{Av}_{0} = b,$$

the inhom. equation again.

Thus:

$Ax = b$ (i) has a unique solution iff $Av = 0$ only has the solution $v = 0$

(ii) Otherwise: $Ax = b$ (a) may not have a solution or

(b) has many solutions that form a line, plane, or hyperplane.

Form "augmented matrix" $(A \mid b)$

Elementary row operations:

(a) re-order rows (change order of equations)

(b) multiply row with a non-zero scalar (multiply equation by a non-zero number)

(c) add a multiple of one row to another (adding/subtracting equations)

Follow example:

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 & 4 \\ -1 & 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right)$$

goal:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right)$$

representing $Ix = b'$

$x = b'$

$$\xrightarrow{\text{re-order rows}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ -1 & 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right) \xrightarrow{R1+R2 \to R2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right)$$

$$\xrightarrow[\substack{R4-2R2 \to R4 \\ R4-R3 \to R4}]{R3-R2 \to R3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\substack{R1-R3 \to R1 \\ R2-R3 \to R2}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 9 \\ 0 & 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$4-(-5)$

Note: $\begin{pmatrix} 9 \\ 11 \\ -5 \\ 0 \end{pmatrix}$ solves $Ax = b$

$\begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ solves $Av = 0$

$$x = \begin{pmatrix} 9 \\ 11 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

a line in 4-dimensional space

$Ax = b$ $\qquad x = A^{-1} b$

"inverse matrix"

not applicable here

span$\{v_1, \dots, v_k\}$ is a vector space, contained in $\vec{V}$