Recall: • $\{v_1, \dots, v_m\}$ l.i. if

$$\underbrace{\alpha_1 v_1 + \dots + \alpha_m v_m}_{} = 0 \quad \text{implies } \alpha_1 = \dots = \alpha_m = 0$$

$$= \sum_{i=1}^{m} \alpha_i v_i \ \text{“=”} \ \sum_i \alpha_i v_i$$

• $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis of $\bar{V}$ if

(i) $\forall x \in \bar{V} \ \exists \alpha_1, \dots, \alpha_n$ s.t. $x = \sum_i \alpha_i b_i$

or: $\bar{V} = \text{span}\{b_1, \dots, b_n\}$

set of all linear combinations of the indicated vectors

we say "$b_1, \dots, b_n$ span $\bar{V}$"

(ii) This representation is unique.

Theorem: $\mathcal{B}$ is a basis $\Rightarrow$ $\mathcal{B}$ are l.i.

Proof: Suppose not. Then there exist $\alpha_1, \dots, \alpha_n$ not all zero s.t. $0 = \sum_i \alpha_i b_i$

But also: $0 = 0 \cdot b_1 + \dots + 0 \cdot b_n$

So this contradicts uniqueness of representation of 0.

Fact: (Proof for later): If V has basis $\mathcal{B}$ that is finite, then every other basis has the same number of vectors n. We say that the dimension of V is n, or $\boxed{\dim V = n}$

Linear operators (transformations) on vector spaces

$A : \bar{V} \to \bar{V}$ is linear if

(i) $A(v+u) = Av + Au \qquad \forall v,u \in \bar{V}$

(ii) $A(\lambda v) = \lambda A v \qquad \forall \lambda \in \mathbb{R}, v \in \bar{V}$

Note: $y = mx + b$ is only linear in this sense if $b = 0$

otherwise we say "affine linear"

Take a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of $\bar{V}$, $v \in \bar{V}$ with $v = \sum_j \alpha_j b_j$

$$\Rightarrow Av = \sum_j \alpha_j A b_j$$

Now suppose $Av = \sum_i \beta_i b_i$

$$A b_j = \sum a_{ij} b_i$$

$$\Rightarrow \sum_i \beta_i b_i = \sum_{i,j} a_{ij}\alpha_j b_i$$

$$\Rightarrow \beta_i = \sum_j a_{ij}\alpha_j \qquad \text{by uniqueness of representation}$$

$$\beta = \begin{pmatrix}\beta_1\\ \vdots \\ \beta_n\end{pmatrix} \qquad \alpha = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} \qquad A = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}}_{\text{matrix}}$$

Introduce the notation

$$\beta = A\alpha$$

matrix multiplication

Examples: V = space of polynomials of degree ≤ 2

$$B = \{1, x, x^2\}$$

$$\mathcal{A} = \frac{d}{dx} \qquad \underbrace{\mathcal{A}1 = 0}_{} \qquad \mathcal{A}x = 1 \qquad \underbrace{\mathcal{A}x^2 = 2x}_{}$$

$$\mathcal{A}1 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 \qquad \mathcal{A}x = 1\cdot 1 + 0\cdot x + 0\cdot x^2 \qquad \mathcal{A}x^2 = 0\cdot 1 + 2\cdot x + 0\cdot x^2$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$p = 3x^2 - 2x + 7 \qquad \alpha = \begin{pmatrix}7\\-2\\3\end{pmatrix}$$

$$\frac{dp}{dx} = 6x - 2 \qquad \beta = \begin{pmatrix}-2\\6\\0\end{pmatrix}$$

$$A\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}7\\-2\\3\end{pmatrix} = \begin{pmatrix} 7\cdot 0 - 2\cdot 1 + 3\cdot 0 \\ 7\cdot 0 - 2\cdot 0 + 3\cdot 2 \\ 7\cdot 0 - 2\cdot 0 + 3\cdot 0\end{pmatrix} = \begin{pmatrix}-2\\6\\0\end{pmatrix}$$

Note: Can take basis $C = \{x^2 - 1, x^2 + 1, x\}$

To represent p in this basis, need to write:

$$\underbrace{\alpha_1(x^2-1) + \alpha_2(x^2+1) + \alpha_3 x}_{} = 3x^2 - 2x + 7$$

$$(\alpha_1 + \alpha_2)x^2 + (\alpha_2 - \alpha_1)\cdot 1 + \alpha_3 x = 3x^2 - 2x + 7$$

Compare coefficients:

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= 3 \\ \alpha_2 - \alpha_1 &= 7 \end{aligned}\right\} \quad 2\alpha_2 = 10 \quad \Rightarrow \alpha_2 = 5 \quad \Rightarrow \alpha_1 = -2$$

$$\alpha_3 = -2$$

Here: $\alpha = \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix}$

$$\mathcal{A}(x^2-1) = 2x = 0\cdot(x^2-1) + 0\,(x^2+1) + 2\cdot x$$
$$\mathcal{A}(x^2+1) = 2x = 0\cdot(x^2-1) + 0\,(x^2+1) + 2\cdot x$$
$$\mathcal{A}x = 1 = -\tfrac{1}{2}(x^2-1) + \tfrac{1}{2}(x^2+1) + 0\cdot x$$

$$A = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 2 & 2 & 0 \end{pmatrix} \qquad A\alpha = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix}$$

as a vector, this represents $1\cdot(x^2-1) - 1\,(x^2+1) + 6\cdot x = 6x - 2$

Correspondence between Operators | Matrices

$$(\mathcal{A}+\mathcal{B})v = \mathcal{A}v + \mathcal{B}v \qquad (A+B)_{ij} = a_{ij} + b_{ij}$$
$$\mathcal{A}(\lambda v) = \lambda\mathcal{A}v \qquad (\lambda A)_{ij} = \lambda a_{ij}$$

$\Rightarrow$ $\mathcal{L}(V)$, the set of linear operators on V, is a vector space

$M(n\times n)$ is a vector space

$$(\mathcal{A}\mathcal{B})v = \mathcal{A}(\mathcal{B}v) \qquad AB \text{ where } \underline{(AB)_{ij} = \sum_k a_{ik} b_{kj}}$$

Makes sense for any $A \in M(n\times m)$, $B \in M(m\times p)$

Example:

$$\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 & 4 \\ 1 & 2 & 1 & 5 \\ 1 & 0 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 2+1+3 & 2+2+0 & -2+1+6 & 8+5+18 \\ -1+0+2 & -1+0+0 & 1+0+4 & -4+0+12 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 4 & 5 & 31 \\ 1 & -1 & 5 & 8 \end{pmatrix}$$

$BA \neq AB$ in general !!!