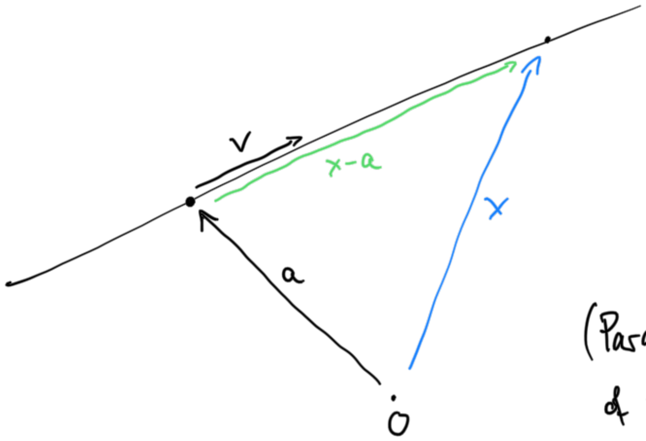
Equations of lines and planes



$$x = a + \lambda v \,, \quad \lambda \in \mathbb{R}$$

(Parametric representation of the line)

Examples: Find the equation of the line through a and b:

$|u \times v| = |u|\,|v| \sin\theta$

a: reference point

$v = b - a$

$\Rightarrow \quad x = a + \lambda(b-a)$ (other choices are possible ...)

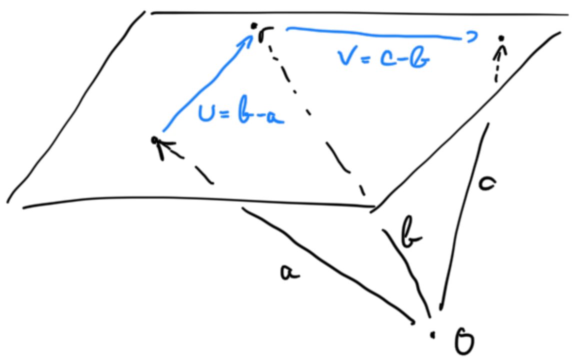
Note:

- x is on the line if $(x-a) \parallel v$

  $\Leftrightarrow (x-a) \parallel (b-a)$

  $\Leftrightarrow (\vec{x}-\vec{a}) \times (\vec{b}-\vec{a}) = 0$

- Any set of two (independent) linear equations in three dimensions defines a line.

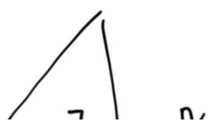


General point on the plane:

$$x = a + \lambda u + \mu v$$

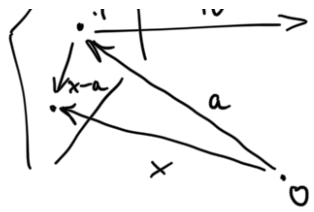
$$\lambda, \mu \in \mathbb{R}$$

"normal equation" of a plane:



$(x-a) \perp n$

$\Leftrightarrow (x-a) \cdot n = 0$



From parametric representation to normal representation: Set $n = u \times v$

From an equation to the normal representation:

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta$$

$$n = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \qquad \beta = a \cdot n \qquad \text{e.g.: } a = \beta \frac{n}{|n|^2}$$

$$\Rightarrow a \cdot n = \beta \cancel{\frac{n \cdot n}{|n|^2}}$$

<u>Example:</u> Find the direction of the line of intersection of the planes defined by

$$x_1 + 3x_2 - x_3 = 5 \qquad \Rightarrow \qquad n = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

$$2x_1 - 2x_2 + 4x_3 = 3 \qquad \Rightarrow \qquad m = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$$

$$v = n \times m = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \cdot 4 - (-1)(-2) \\ (-1) \cdot 2 - 1 \cdot 4 \\ 1 \cdot (-2) - 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -6 \\ -8 \end{pmatrix}$$

<u>Vector spaces</u>

V is a vector space over $\mathbb{R}$ (or $\mathbb{C}$) if

(i) V is closed under commutative and associative addition, i.e.

$$+ : V \times V \to V$$

$$\text{with} \quad \left.\begin{aligned} a + b &= b + a \\ (a+b) + c &= a + (b + c) \end{aligned}\right\} \forall a, b, c \in V$$

(ii) V is closed under distributive and associative scalar multiplication, i.e.

$$\cdot : \mathbb{R} \times V \to V$$

$$\text{with} \quad \left.\begin{aligned} \lambda(a+b) &= \lambda a + \lambda b \\ (\lambda + \mu) a &= \lambda a + \mu a \\ \lambda(\mu a) &= (\lambda \mu) a \end{aligned}\right\} \forall a, b \in V \quad \forall \lambda, \mu \in \mathbb{R}$$

(iii) $\exists\, 0 \in V$ s.t. $a + 0 = a \quad \forall a \in V$

(iii) $\exists\, 0 \in V$ ...

(iv) $1 \cdot a = a \quad \forall a \in V$

<u>Note:</u>

- 0 is unique: $$0 + 0 = 0$$ $$0 + 0 = 0$$

- For every $a \in V$ there is $-a := (-1)a \quad \Rightarrow \quad a - a = 0$

  indeed: $a - a = \underbrace{(1-1)}_{0} a = 0$

<u>Examples:</u>

- $V = \mathbb{R}^n$
- $V = \{p : p \text{ polynomial of degree} \leq n\}$
- $V = \{\text{continuous functions on } [0,1]\}$

<u>Non-examples:</u>

- $V = \{p : p \text{ polynomial of degree} = n\}$
- images with grayscale values $0 \dots 255$
- probability vectors: $p = (p_1, \dots, p_n) \qquad p_i \in [0,1]$, $p_1 + \dots + p_n = 1$
- $\mathbb{R}^3$ where "addition" is the cross product.

$\{v_1, \dots, v_m\}$ is <u>linearly independent</u> if

$$\underbrace{\alpha_1 v_1 + \dots + \alpha_m v_m}_{\text{"linear combination"}} = 0 \quad \text{implies} \quad \alpha_1 = \dots = \alpha_m = 0$$

$\{b_1, \dots, b_n\}$ is called a <u>basis</u> of V if

(i) For every $x \in V$ there are $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ s.t. $x = \alpha_1 b_1 + \dots + \alpha_n b_n$

(ii) this representation is unique.

If B is a basis of V, the vector $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n$ is called the coordinate vector of x.