Analytic Geometry

Vectors in Euclidean space $\mathbb{R}^n$, $n = 2, 3$ mostly

- quantity with <u>magnitude</u> and <u>direction</u>
- can be thought of specifying a position or displacement, but not both

"displacement of the origin"

- represented by coordinates

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

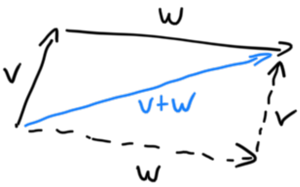
"components" or "entries"

"column vector" - default

- notation: $\vec{v}$, $\underline{v}$, boldface .... here: no special indication! (Context)

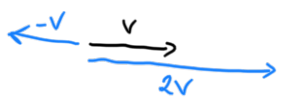
- Two basic operations:

$$v + w = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$$



addition

$$\lambda v = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}$$

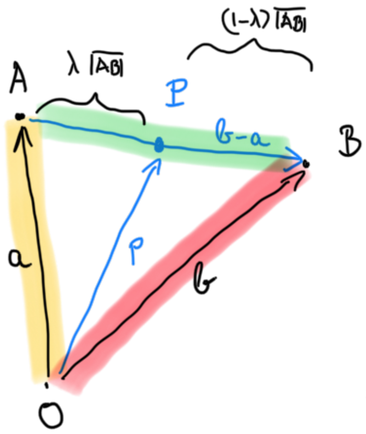


scalar multiplication

$\lambda \in \mathbb{R}$ (or $\mathbb{C}$) "scalar"

These operations obey the usual rules of arithmetic (more formal statements for later)

<u>Examples:</u> ① Point on a line segment



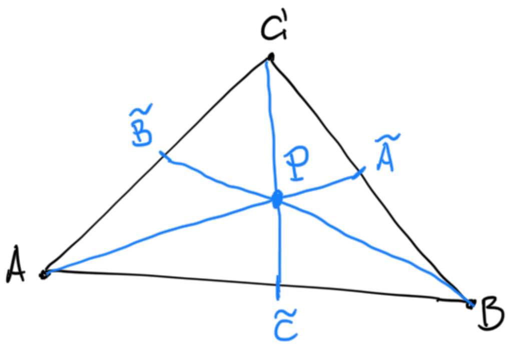
$$a + (b - a) = b$$

Suppose P is located a fraction λ of the length of the line segment from A.

$$\Rightarrow p = a + \lambda(b - a)$$
$$= (1-\lambda) a + \lambda b$$

for $\lambda \in [0,1]$, this is called a "convex combination"

② Centroid of a triangle



P: centroid

Q: · Does this construction really give a single point of intersection?

· Coordinates of P?

Coordinates of $\tilde{A}, \tilde{B}, \tilde{C}$:
$$\tilde{a} = \tfrac{1}{2}b + \tfrac{1}{2}c$$
$$\tilde{b} = \tfrac{1}{2}a + \tfrac{1}{2}c$$
$$\tilde{c} = \tfrac{1}{2}a + \tfrac{1}{2}b$$

Line segment $\overline{A\tilde{A}}$: $\lambda a + (1-\lambda)\tilde{a}$

" " $\overline{B\tilde{B}}$: $\mu b + (1-\mu)\tilde{b}$

" " $\overline{C\tilde{C}}$: $\varkappa c + (1-\varkappa)\tilde{c}$

$$\lambda a + (1-\lambda)\tfrac{1}{2}b + (1-\lambda)\tfrac{1}{2}c \overset{(*)}{=} \mu b + (1-\mu)\tfrac{1}{2}a + (1-\mu)\tfrac{1}{2}c$$

To solve (*), let's equate coefficients in front of a, b, c separately.

$$\lambda = \frac{1-\mu}{2} \qquad \frac{1-\lambda}{2} = \mu \qquad \frac{1-\lambda}{2} = \frac{1-\mu}{2}$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\text{consistent} \Leftarrow \quad \frac{1}{2} - \frac{\lambda}{2} = \lambda \quad \Leftarrow \quad \lambda = \mu$$
$$\Rightarrow \lambda = \frac{1}{3}$$

$$p = \tfrac{1}{3}a + \left(1-\tfrac{1}{3}\right)\tfrac{1}{2}b + \left(1-\tfrac{1}{3}\right)\tfrac{1}{2}c = \tfrac{1}{3}(a+b+c)$$

Using $\overline{B\tilde{B}}$ and $\overline{C\tilde{C}}$ to do this computation will give the same point by symmetry.

<u>Magnitude of a vector:</u> (or length)
$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Unit vector in the direction of $v \neq 0$: "encodes only the direction information"
$$\hat{v} = \frac{v}{|v|} \qquad \Rightarrow \quad |\hat{v}| = 1$$

$|v|$

$v = |v|\,\hat{v}$ "polar decomposition"

The <u>scalar product</u> (or "inner product" or "dot product")

$u \cdot v = |u|\,|v| \cos\theta$ θ: angle between u and v

$= u_1 v_1 + u_2 v_2 + u_3 v_3$

$= u^T v$ (for later)

note: $u \cdot v = 0$, then u is perpendicular to v, $(u \perp v)$

(with the understanding that $0 \perp v$ for any $v \in \mathbb{R}^n$)

Remark: If u, v have complex entries, then

$u \cdot v = u_1^* v_1 + u_2^* v_2 + \dots = u^H v$

then $u \cdot v = (v \cdot u)^*$

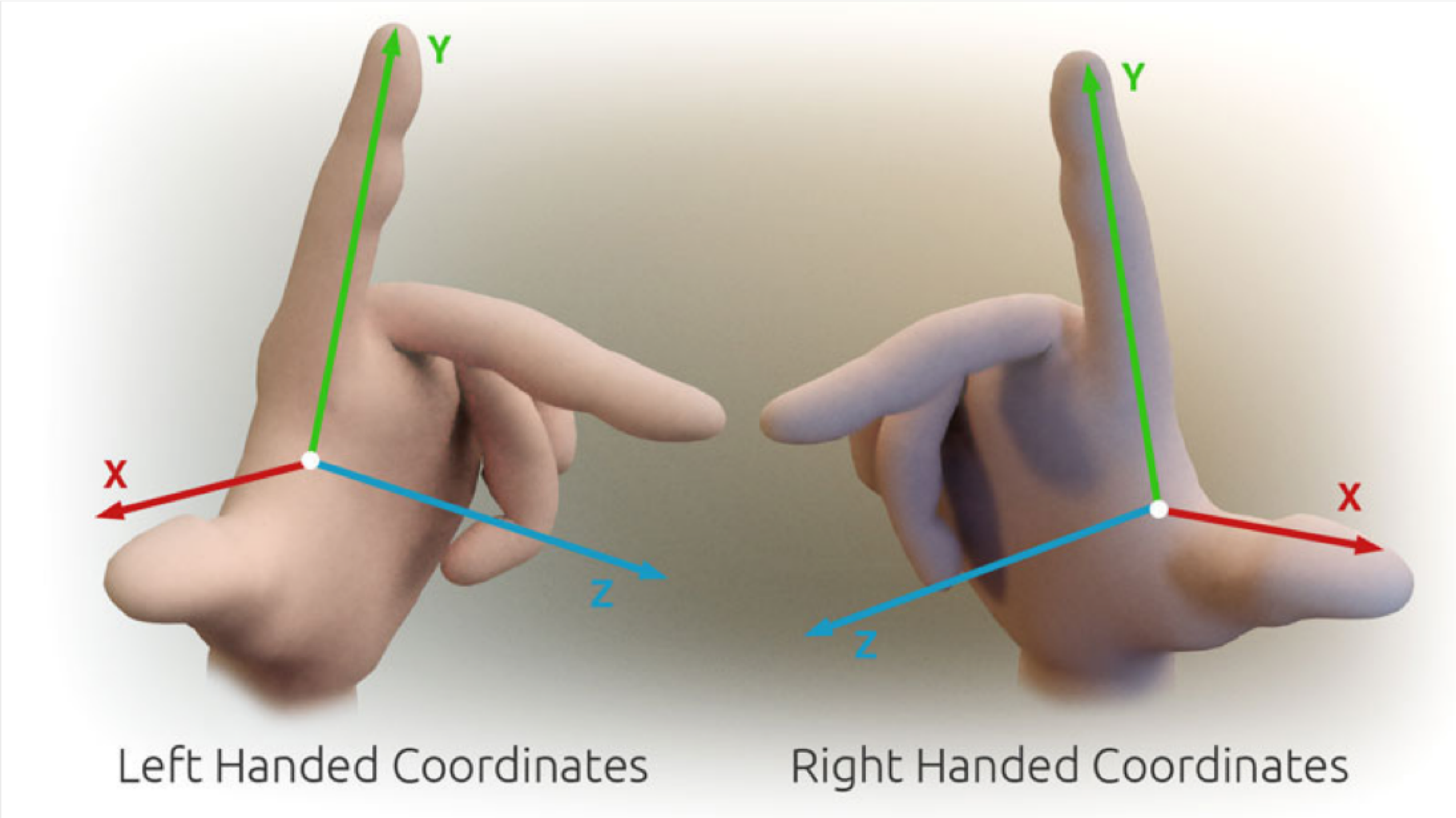
In all cases: $|u|^2 = u \cdot u$

The <u>cross product</u> in $\mathbb{R}^3$ is defined

$|u \times v| = |u|\,|v| \sin\theta$

in the direction that is perpendicular to u and v with the convention that

$u \times v$, u, v are a right-handed system



Coordinate expression:

$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

<u>Properties:</u>

- $(u+v) \times w = u \times w + v \times w$
- $u \times v = - v \times u$
- $u \times (v \times w) \neq (u \times v) \times w$