Recall: Model for limited growth

$$\frac{dI}{dt} = r I \left(1 - \frac{I}{K}\right) \qquad \text{"Logistic equation"}$$

$$I(0) = I_0$$

$$\int_{I_0}^{I(t)} \underbrace{\frac{dI}{I(1-\frac{I}{K})}}_{\left(\frac{1}{I} + \frac{1}{K-I}\right) dI} = \int_0^t r\,dt \quad \Rightarrow \quad \ln I - \ln(K-I) \Big|_{I_0}^{I(t)} = rt$$

$$\Rightarrow \quad \ln \frac{I}{K-I} \Big|_{I_0}^{I(t)} = rt$$

$$\Rightarrow \quad \ln\left(\frac{I(t)}{K-I(t)} \; \frac{K-I_0}{I_0}\right) = rt$$

$$\Rightarrow \quad \frac{I(t)}{K-I(t)} = \underbrace{\frac{I_0}{K-I_0} e^{rt}}_{=:\alpha}$$

$$\frac{I}{K-I} = \alpha \Rightarrow I = \alpha K - \alpha I \Rightarrow I(1+\alpha) = \alpha K \Rightarrow I = \frac{\alpha}{1+\alpha} K = \frac{1}{\frac{1}{\alpha}+1} K$$

$$I(t) = \frac{1}{\frac{K-I_0}{I_0} e^{-rt} + 1} K = \frac{K I_0}{\underbrace{(K-I_0) e^{-rt}}_{\to 0 \text{ as } t \to \infty} + I_0}$$

- $\lim\limits_{t\to\infty} I(t) = K$
- $I(0) = \dfrac{K I_0}{K - I_0 + I_0} = I_0$

Recall initial differential equation

$$\frac{dI}{dt} = r I \left(1 - \frac{I}{K}\right)$$

roots of RHS: $I=0$, $I=K$

So: if $I_0 = 0$ then $\frac{dI}{dt} = 0 \Rightarrow I(t) = 0$ for all $t \geq 0$.

$I_0 = K$ then ------ $I(t) = K$ -----

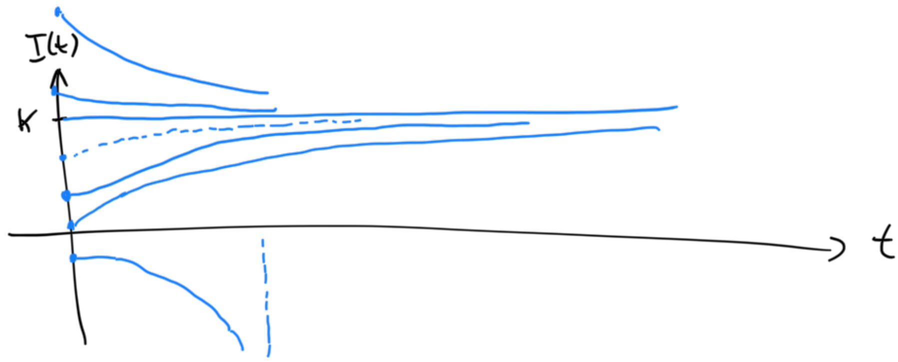
Points where the rate of change is zero are called equilibrium points.

When $I_0 > K$: $\frac{dI}{dt} < 0 \Rightarrow I$ is decreasing

$I_0 \in (0, K)$: $\frac{dI}{dt} > 0 \Rightarrow I$ is increasing

$I_0 < 0$: $\frac{dI}{dt} < 0 \Rightarrow I$ is decreasing

Fact: Solutions are unique: two solution curves cannot cross.



More examples:

① Newton's law of cooling

$$\frac{dT}{dt} = k(A - T) \qquad k > 0 \quad \text{"rate coefficient"}$$

A "ambient temperature"

$$T(0) = T_0$$

$$\int_{T_0}^{T(t)} \frac{dT}{A-T} = \int_0^t k\,dt$$

$$\Rightarrow -\ln(A-T)\Big|_{T_0}^{T(t)} = kt$$

$$\Rightarrow \ln \frac{A-T_0}{A-T(t)} = kt$$

$$\Rightarrow \frac{A-T_0}{A-T(t)} = e^{kt}$$

$$\Rightarrow A - T(t) = (A-T_0)\,e^{-kt}$$

$$\Rightarrow \quad T(t) = A + (T_0 - A)\,e^{-kt} \xrightarrow[t\to\infty]{} A$$

check: $T(0) = A + (T_0 - A)\cdot 1 = T_0$

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② $\dfrac{dy}{dx} = -3xy \qquad y(0) = 1 \qquad\qquad y = y(x)$

$$\int_1^{y(x)} \frac{dy}{y} = -\int_0^x 3\tilde{x}\, d\tilde{x}$$

$$\Rightarrow \quad \ln y \Big|_1^{y(x)} = -\frac{3}{2}x^2 \Big|_0^x$$

$$\Rightarrow \quad \ln y(x) = -\frac{3}{2}x^2$$

$$\Rightarrow \quad y(x) = e^{-\frac{3}{2}x^2}$$

General pattern: "separable differential equations":

$$y' = g(x)\,h(y)$$

$$\Rightarrow \quad \int \frac{dy}{h(y)} = \int g(x)\,dx$$

Solve integrals, then solve for $y(x)$.

③ $\dfrac{dy}{dx} = y^2 \qquad y(1) = 1$

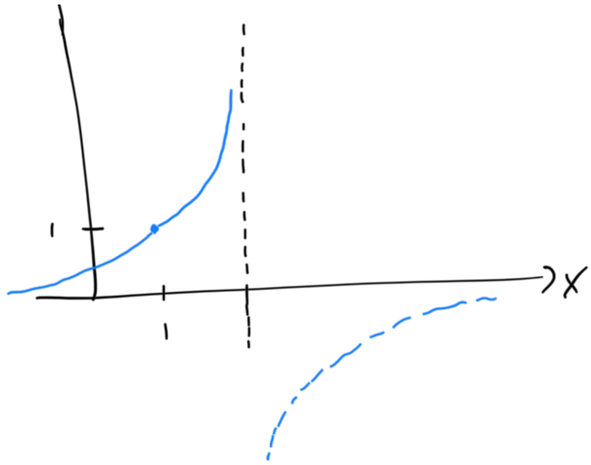
$$\int_1^{y(x)} \frac{dy}{y^2} = \int_1^x dx \qquad \Rightarrow \quad -y^{-1}\Big|_1^{y(x)} = x\Big|_1^x$$

$$\Rightarrow \quad 1 - \frac{1}{y(x)} = x - 1$$

$$\Rightarrow \quad \frac{1}{y(x)} = 2 - x$$

$$\Rightarrow \quad y(x) = \frac{1}{2-x}$$

"blow-up" at $x = 2$



④ $\frac{dy}{dx} = \sqrt{y} \qquad y(0) = y_0$

$$\int_{y_0}^{y(x)} y^{-\frac{1}{2}}\, dy = \int_0^x dx \Rightarrow 2y^{\frac{1}{2}} \Big|_{y_0}^{y(x)} = x \Rightarrow \sqrt{y(x)} = \frac{x}{2} + \sqrt{y_0}$$

$$\Rightarrow y(x) = \left(\frac{x}{2} + \sqrt{y_0}\right)^2$$

Check: $\frac{dy}{dt} = \frac{2x}{4} = \frac{x}{2}$

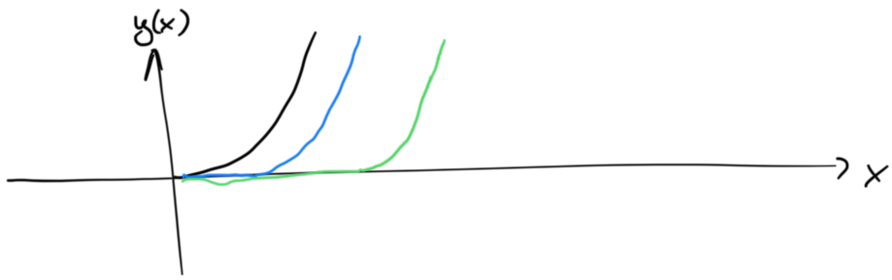
$\sqrt{y} = \sqrt{\frac{x^2}{4}} = \frac{x}{2}$

$y(0) = 0$

In particular: if $y_0 = 0$: $\quad y(x) = \frac{x^2}{4}$

Note: there is another solution: $y(x) = 0$

Violation of uniqueness !!!



⑤ Volterra-Lotka equations

$\frac{dy}{dt} = by - \tau x y$ Prey

$$\frac{dx}{dt} = -sx + cxy$$ Predator

Equation for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{by - rxy}{-sx + cxy} = \frac{(b - rx)\, y}{x\,(-s + cy)}$$

$$\Rightarrow \int \underbrace{\frac{-s+cy}{y}}_{-\frac{s}{y}+c} dy = \int \underbrace{\frac{b - rx}{x}}_{\frac{b}{x}-r} dx$$

$$\Rightarrow -s \ln y + cy = b \ln x - rx + C'$$

Equilibrium points: $x = y = 0$ (not interesting) or

$b - rx = 0$ and $s - cy = 0$ $\Rightarrow$ $x = \frac{b}{r}$ $y = \frac{s}{c}$