Recall: work

$$W = \int_a^b F(x)\,dx$$

① Particle of mass m is moving along a trajectory $x(t)$

position $x(t)$
velocity $v(t) = \frac{dx}{dt}$
acceleration $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Newton's second law $F = ma$

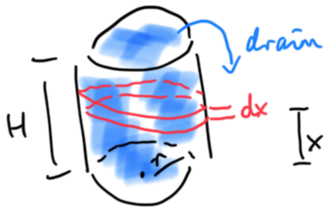
$$\Rightarrow W = m \int_a^b \frac{dv}{dt}\,dx \qquad dx = \frac{dx}{dt}\,dt = \underbrace{v\,dt}$$

$$= m \int_{t_a}^{t_b} \underbrace{\frac{dv}{dt}\,v}_{=\frac{1}{2}\frac{d}{dt}v^2}\,dt$$

where t_a, t_b satisfy $x(t_a) = a$ and $x(t_b) = b$

$$= \frac{m}{2} \int_{t_a}^{t_b} \frac{d}{dt}(v^2)\,dt = \frac{m}{2} v^2 \Big|_{t_a}^{t_b} = \frac{m}{2} v^2(t_b) - \frac{m}{2} v^2(t_a)$$

Conclusion: work is change in (kinetic) energy.

② 

drain out over top of barrel

Work required to lift red layer over the top is the change in potential energy:

$$dE = g\,\underbrace{(H-x)}_{\text{height that layer needs to be lifted}}\,dm \qquad E_{pot} = mgh$$

$$= g(H-x)\,\underset{\substack{\uparrow\\ \text{density}}}{\varrho} \cdot A \cdot dx$$

$A = \pi r^2$, cross-sectional area of the slice.

$$W = \int dE = \int^H g(H-x)\,\varrho \cdot \pi r^2\,dx = g\varrho\pi r^2 \int^H (H-x)\,dx$$

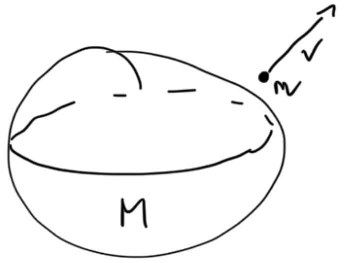
$\dots \qquad 0$

$$= \frac{1}{2}\pi r^2 g 3\, H^2$$

$$\overbrace{\phantom{Hx - \frac{1}{2}x^2}}^{0}$$

$$= Hx - \frac{1}{2}x^2 \Big|_0^H = H^2 - \frac{1}{2}H^2 - (0-0)$$
$$= \frac{1}{2}H^2$$

③ Escape velocity



Q: what initial velocity does the object of mass m require to "escape" from the gravitational field of the planet of mass M and radius R.

Need Newton's law of gravitation: $F(r) = \frac{G m M}{r^2}$

← gravitational constant

↖ distance of centers of the bodies

Work required to send object from surface to ∞ :

$$\bar{W} = \int_R^\infty F(r)\, dr = G m M \underbrace{\int_R^\infty \frac{dr}{r^2}}_{-r^{-1}\big|_R^\infty \, = \, \frac{1}{R}} = \frac{G m M}{R}$$

If initial velocity of object is v, it has kinetic energy $K = \frac{1}{2} m v^2$

For escape velocity, this should coincide with $\bar{W}$, so

$$\frac{G \cancel{m} M}{R} = \frac{1}{2} \cancel{m} v_{esc}^2$$

$$\Rightarrow v_{esc} = \sqrt{\frac{2GM}{R}}$$

Differential equations: here seek function y(t) which satisfies an equation of the form

$$\frac{dy}{dt} = f(y(t), t)$$

Examples:

① Simple growth of number of infected individuals I during an epidemic:

rate coefficient for recovering

$$\frac{dI}{dt} = \beta I - \gamma I \qquad I(0) = I_0 \quad \text{initial number infected}$$

transmission rate coefficient

$$= r I \qquad r = \beta - \gamma$$

$r > 0$: number of infected is increasing

$r < 0$: # of infected is decreasing

$$\int_{I_0}^{I(t)} \frac{dI}{I} = \int_0^t r \, dt$$

$$\Rightarrow \underbrace{\ln I \Big|_{I_0}^{I(t)}}_{\ln I(t) - \ln I_0 = \ln \frac{I(t)}{I_0}} = rt \qquad \Rightarrow \frac{I(t)}{I_0} = e^{rt}$$

$$\Rightarrow I(t) = I_0 e^{rt}$$

Doubling time T_2, time it takes for I to double:

$$2 = e^{rT_2} \quad = \boxed{T_2 = \frac{\ln 2}{r}}$$

(If $r < 0$, we speak of "half-life", e.g. radioactive decay.)

② Limited growth

$$\frac{dI}{dt} = \beta \underbrace{\left(1 - \frac{I}{P}\right)}_{\text{fraction of population still available for infection}} I - \gamma I = (\beta - \gamma) I - \frac{\beta}{P} I^2 \qquad P \text{ is population size}$$

$$= r I \left(1 - \frac{I}{K}\right) = r I - \frac{r}{K} I^2 \qquad \frac{\beta}{P} = \frac{r}{K} \quad \text{or } K = \frac{rP}{\beta}$$

$$\frac{dI}{I(1-\frac{I}{K})} = r\,dt \qquad \Rightarrow \int_{I_0}^{I(t)} \frac{dI}{I(K-I)} = \underbrace{\int_0^t \frac{r}{K}\,dt}_{\frac{r}{K}t}$$

For left-hand integral, need partial fractions:

$$\frac{1}{I(K-I)} = \frac{A}{I} + \frac{B}{K-I} = \frac{AK - AI + BI}{I(K-I)}$$

$$\Rightarrow B = A \qquad AK = 1 \Rightarrow A = \frac{1}{K}$$

$$\Rightarrow \underbrace{\int_{I_0}^{I(t)} \left(\frac{1}{I} + \frac{1}{K-I}\right) dI}_{\ln I - \ln(K-I)\Big|_{I_0}^{I(t)} = \ln\frac{I}{K-I}\Big|_{I_0}^{I(t)} = \ln\left(\frac{I(t)}{K-I(t)}\cdot\frac{K-I_0}{I_0}\right)} = rt$$

$$\Rightarrow \frac{I(t)}{K-I(t)} = \frac{I_0}{K-I_0}\,e^{rt}$$

$$\Rightarrow \text{-----} \Rightarrow I(t) = \frac{I_0 K}{(K-I_0)e^{-rt}+I_0} \xrightarrow[t\to\infty]{} K$$

K: "carrying capacity of enviroment"