Theorems on definite integration

$f, g : [a,b] \to \mathbb{R}$ integrable

(i) if $f \geq 0$, then $\int_a^b f(x)\,dx \geq 0$

(ii) $f \geq g$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ From (i), use $f-g$ ....

(iii) $$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$ From (i), split f into pos. and negative part ....

(iv) If f is continuous, then there exists a $\xi \in [a,b]$ s.t.

$$\underbrace{\frac{1}{b-a} \int_a^b f(x)\,dx}_{\text{average of } f \text{ on the interval } [a,b]} = f(\xi)$$

"Integral Mean Value Theorem"

Proofs: (i) Recall: $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=0}^{n-1} \underbrace{f(x_i)}_{\geq 0} \underbrace{\Delta x}_{\geq 0}}_{\geq 0}$$ $x_i = a + i\Delta x$, $\Delta x = \frac{b-a}{n}$

Result follows by properties of the limit.

(iv) $m = \min_{x\in[a,b]} f(x)$ (exists, because f cont. on a closed bounded interval)

$M = \max_{x\in[a,b]} f(x)$ "

$\Rightarrow m \leq f(x) \leq M$

$$\overset{(ii)}{\Rightarrow} \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$m(b-a)$ $\qquad\qquad$ $M(b-a)$

$$\Rightarrow \quad m \leq \underline{\frac{1}{b-a}\int_a^b f(x)\,dx} \leq M$$

Now f is cont., so by the IVT, f takes every value between m and M, so in particular, there exists $\xi \in [a,b]$ s.t.

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad \square$$

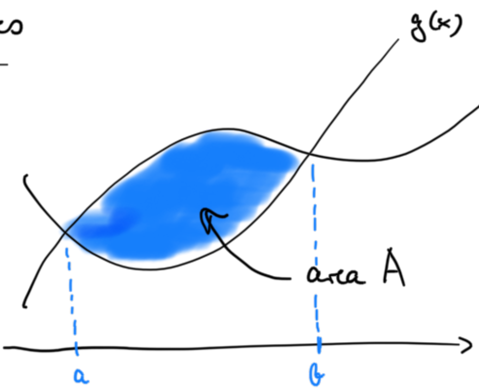
Remark: If F is an antiderivative of f, the FTC says that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{F(b)-F(a)}{b-a} \underset{\substack{\uparrow \\ \text{MVT of differential calc.}}}{=} F'(\xi) \quad \text{for some } \xi \in (a,b)$$

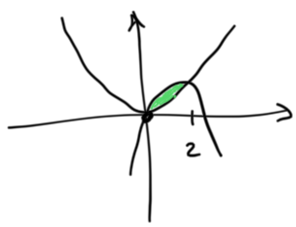
$$= f(\xi)$$

This shows that the mean value theorems of diff. calc. and integral calculus are related by the FTC.

Area between curves



Task: compute A



Examples:

① $y = x^2$
$y = 6x - 2x^2$

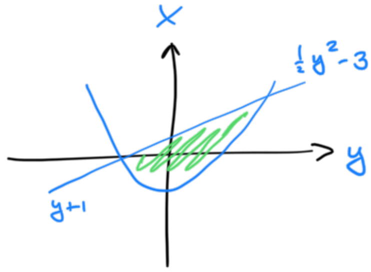
Find points of intersection:

$x^2 = 6x - 2x^2 \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

$a = 0 \qquad b = 2$

$$A = \int_0^2 (6x - 2x^2) - x^2 \, dx = \int_0^2 (6x - 3x^2) dx = 3x^2 - x^3 \Big|_0^2 = 3 \cdot 4 - 2 \cdot 4 = 4$$

② $y^2 = 2x + 6 \quad \Rightarrow \quad x = \frac{1}{2}y^2 - 3$
$y = x - 1 \quad \Rightarrow \quad x = y + 1$



Points of intersection, treat y as the independent variable:

$$\frac{1}{2}y^2 - 3 = y + 1 \Rightarrow y^2 - 2y - 8 = 0$$

use quadratic formula (or guessing) to find that the roots are $-2, 4$

$$A = \int_{-2}^{4} (y+1) - (\tfrac{1}{2}y^2 - 3) \, dy = \int_{-2}^{4} (-\tfrac{1}{2}y^2 + y + 4) \, dy$$

$$= -\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y \Big|_{-2}^{4} = -\frac{1}{6} 64 + \frac{1}{2} \cdot 16 + 4 \cdot 4 - \left(\frac{1}{6} \cdot 8 + \frac{1}{2} \cdot 4 - 2 \cdot 4\right)$$

$$= \dots = 18$$

Integration of rational functions

Example: ① $\int \frac{1}{1+x^2} dx = \arctan x + C$

② $\int \frac{1}{x} dx = \ln|x| + C$

③ $\int \frac{x^3 + x}{x^2 - 1} dx$

$$(x^3 + x) : (x^2 - 1) = x + \frac{2x}{x^2 - 1}$$
$$\underline{-(x^3 - x)}$$
$$2x$$

$$= \int x \, dx + \int \frac{2x}{x^2 - 1} dx$$

$u = x^2 - 1 \qquad \frac{du}{dx} = 2x \Rightarrow 2x \, dx = du$

$$= \frac{1}{2}x^2 + \int \frac{du}{u}$$

$$= \frac{1}{2}x^2 + \ln|x^2 - 1| + C$$

$\underbrace{\qquad}_{|u|}$

In general, a rational function, i.e., a ratio of polynomials $\frac{N(x)}{D(x)}$, can be written as a "linear combination" of the following terms:

(i) A polynomial of degree $(\deg N - \deg D)$

(ii) rational functions of the form $\frac{A_1}{x-x_i}, \frac{A_2}{(x-x_i)^2}, \dots \frac{A_k}{(x-x_i)^k}$

where x_i is a root of D of multiplicity k

(iii) rational functions of the form

$$\frac{A_1+B_1x}{ax^2+bx+c}, \quad \frac{A_2+B_2x}{(ax^2+bx+c)^2}, \dots \frac{A_m+B_mx}{(ax^2+bx+c)^m}$$

If $(ax^2+bx+c)^m$ is a factor of D

<u>Remark:</u> If we are willing to use complex numbers, cases (i) and (ii) are enough.

<u>Example:</u> $\displaystyle\int \frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}\,dx$

$$\begin{array}{l} (4x^3+23x^2+45x+27) : (x^3+5x^2+8x+4) = 4 \\ \underline{-\,]\,4x^3+20x^2+32x+16} \\ \qquad 3x^2+13x+11 \end{array}$$

$$= \int \left(4 + \frac{3x^2+13x+11}{x^3+5x^2+8x+4}\right) dx = I$$

Roots of $D(x)$: $x=-1$ is a root (by guessing), divide it out:

$$(x^3+5x^2+8x+4) : (x+1) = \underbrace{x^2+4x+4}_{=(x+2)^2}$$

$$\begin{array}{l} \underline{-\,]\,x^3+x^2} \\ \qquad 4x^2+8x \\ \qquad \underline{-\,]\,4x^2+4x} \\ \qquad\qquad 4x+4 \\ \qquad\qquad \underline{-\,]\,4x+4} \\ \qquad\qquad\quad 0 \end{array}$$

$\Rightarrow D(x) = (x+1)(x+2)^2$

This means we should expect a partial fraction decomposition of the form

$$\frac{3x^2+13x+11}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \qquad \text{for some } A, B, C$$

match these expressi

$$= \frac{A(x+2)^2 + B(x+1)(x+2) + C(x+1)}{(x+1)(x+2)^2}$$

For $x=-1$: $\quad \underbrace{3-13+11}_{1} = A\underbrace{(-1+2)}_{1}^2 + B\cdot 0 + C\cdot 0 \quad \Rightarrow \quad A = 1$

$x=-2$: $\quad \underbrace{3\cdot 4 - 13\cdot 2 + 11}_{-3} = A\cdot 0 + B\cdot 0 + C\underbrace{(-2+1)}_{=-1} \quad \Rightarrow \quad C = 3$

$x=0$: $\quad 11 = 1\cdot 2^2 + B\cdot 2 + 3\cdot 1$

$\quad 4 = 2B \quad \Rightarrow \quad B = 2$

$$\Rightarrow I = \int \left(4 + \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2}\right) dx$$

$$= 4x + \ln|x+1| + 2\ln|x+2| - 3\,\frac{1}{x+2} + C_1$$

$\int \frac{1}{(x+2)^2}\,dx \qquad u = x+2, \quad du = dx$

$= \int \frac{du}{u^2} = -u^{-1} + C_1$

$= -\frac{1}{x+2} + C_1$

Another example:

$$\int \frac{1}{\cos x}\,dx$$

Trick: $u = \sin x$

$du = \cos x\,dx$

$\cos^2 x + \overset{u}{\sin^2 x} = 1$

$\to \cos^2 x = 1 - u^2$

$$= \int \frac{du}{\cos^2 x} = \int \frac{du}{1-u^2} = \int \frac{du}{(1-u)(1+u)}$$

$$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$= \frac{A(1+u) + B(1-u)}{(1-u)(1+u)} \qquad =1$$

$$= \frac{1}{2}\int \frac{1}{1-u}\,du + \frac{1}{2}\int \frac{1}{1+u}\,du$$

$\Rightarrow \left.\begin{array}{l} A = B \\ A + B = 1 \end{array}\right\} A = B = \frac{1}{2}$

$$= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C_1$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C_1$$

$= -\int \frac{1}{u-1}\,du = -\ln|u-1| = -\ln|1-u|$

$$= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C_1$$

$$\int \frac{\quad}{1-u}\, du \qquad v = 1-u$$

$$dv = -du$$

$$= -\int \frac{dv}{v} = -\ln|v| + C_1 = -\ln|1-u| + C_1$$