Recall: $\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx$

Examples: (2) $\int \ln x\,dx = \int \ln x \cdot 1\,dx = (\ln x)\,x - \int \underbrace{\frac{1}{x}\,x}_{1}\,dx$

$$= x\ln x - x + c$$

(3) $\underbrace{\int e^x \sin x\,dx}_{I} = e^x \sin x - \underbrace{\int e^x \cos x\,dx}$

$$= e^x \cos x - \int e^x(-\sin x)\,dx$$

$$= \underbrace{e^x(\sin x - \cos x)}_{J} - \underbrace{\int e^x \sin x\,dx}_{I}$$

$I = J - I$

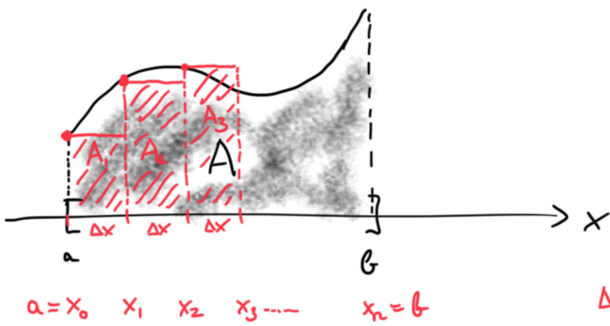
$\Rightarrow 2I = J$

$\Rightarrow I = \frac{1}{2}J$

$$\Rightarrow \int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + c$$

The definite integral

$f : [a,b] \to \mathbb{R}$



$\Delta x = \frac{b-a}{n}$

$$A \approx A_1 + A_2 + \dots + A_n$$

$$= \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

Def: Let $f : [a,b] \to \mathbb{R}$ be cont. except perhaps at a countable number of points, f has a jump discontinuity. Then the definite integral of f over $[a,b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

Remarks: • This limit is known to exist

- It is independent of the choice of evaluation point of f within the interval $[x_{i-1}, x_i]$, and is independent of the partition so long as it gets finer as $n \to \infty$.

- The class of "Riemann-integrable functions" is actually larger than stated

$\longrightarrow$ Analysis I

Examples: ① $\displaystyle\int_a^b 1\,dx = b-a$

② $$\int_0^1 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{x_{i-1}}\,\Delta x \qquad \Delta x = \frac{1}{n}, \quad x_i = 0 + i\Delta x = \frac{i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} e^{\frac{i}{n}}\,\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left(e^{\frac{1}{n}}\right)^i \qquad q = e^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \frac{\left(e^{\frac{1}{n}}\right)^n - 1}{e^{\frac{1}{n}} - 1} \qquad (*)$$

$$= (e-1) \underbrace{\lim_{n\to\infty} \frac{1}{n} \frac{1}{e^{\frac{1}{n}} - 1}}_{\lim\limits_{h\to 0} \frac{1}{\frac{e^h - e^0}{h}} = \frac{1}{\exp'(0)} = \frac{1}{e^0} = 1} \qquad h = \frac{1}{n}$$

$$= e - 1$$

Why is $(*)$ true? This is called a geometric sum, of the form

$$S_n = \sum_{i=0}^{n-1} q^i = q^0 + q^1 + q^2 + \cdots + q^{n-1} \qquad q \in \mathbb{R}$$

$$- \quad q S_n = \qquad q^1 + q^2 + q^3 + \cdots + q^{n-1} + q^n$$

$$S_n - qS_n = 1 - q^n$$

$$\Rightarrow S_n(1-q) = 1 - q^n$$

$$\Rightarrow \boxed{S_n = q^0 + q^1 + \ldots + q^{n-1} = \frac{1-q^n}{1-q}} \qquad q \neq 1$$

<u>Properties of the definite integral:</u>

(i) The integral is linear: $\int_a^b (f+g)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx, \quad c \in \mathbb{R}$$

(ii) $\int_a^a f(x)\,dx = 0$

(iii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Fundamental Theorem of Calculus (FTC)

$f : [a,b] \to \mathbb{R}$ cont.

(i) $F(x) = \int_a^x f(t)\,dt$ is an anti-derivative of f.

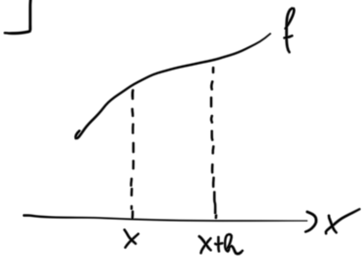
(ii) If F is an anti-derivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

$$\int_a^{b} f(x)\,dx = F(b)-F(a) = F(x)\Big|_a^{b}$$

Proof: (i)

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^{a} f(t)\,dt\right]$$



$$= \underbrace{\frac{1}{h}\int_x^{x+h} f(t)\,dt}_{\text{Average of } f \text{ over the interval } [x,\,x+h]}$$

Detailed discussion using MVT of integration next time.

$$\xrightarrow[h\to 0]{} f(x)$$

$$\Rightarrow F'(x) = f(x)$$

(ii) If F is an arbitrary antiderivative, we know that

$$F(x) = \int_a^{x} f(t)\,dt + c$$

$$F(a) = c$$

$$F(b) = \int_a^{b} f(t)\,dt + c = \int_a^{b} f(t)\,dt + F(a)$$

$$\Rightarrow \int_a^{b} f(t)\,dt = F(b) - F(a)$$

Examples:

① $$\int_1^{2} x\,dx = \frac{1}{2}x^2\Big|_1^{2} = \frac{1}{2}2^2 - \frac{1}{2}1^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

② $$\int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x\Big|_0^{\frac{\pi}{2}} = -\underbrace{\cos\tfrac{\pi}{2}}_{0} + \underbrace{\cos 0}_{1} = 1$$

③ $$\int_{-2}^{-1} \frac{1}{x}\,dx$$

$u = -x \Rightarrow du = -dx$ $\qquad U'(x) = \frac{du}{dx} = -1$

$x = -u \qquad dx = -du$

$$= \int_2^{1} \frac{1}{-u}(-du)$$

Note: Limits of integration must be substituted, too.

$$= \int_2^1 \frac{du}{u} = \ln u \Big|_2^1 = \ln 1 - \ln 2 = -\ln 2$$

(4) $$\int_0^1 x^2 \sin(x^3+1)\, dx$$

$u = x^3 + 1 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow x^2 dx = \frac{du}{3}$

$$= \int_{0^3+1}^{1^3+1} \sin(u) \frac{du}{3} = \frac{1}{3} \int_1^2 \sin u \, du = -\frac{1}{3} \cos u \Big|_1^2$$

$$= \frac{1}{3} (\cos 1 - \cos 2)$$