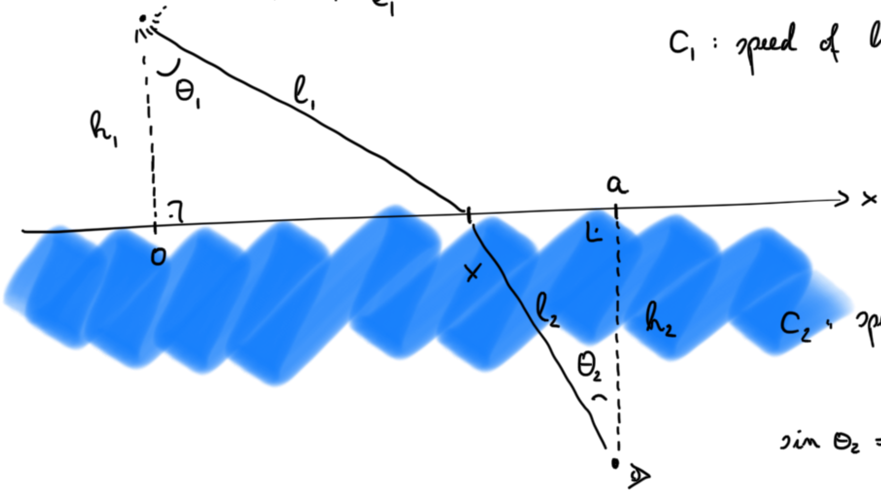
Application: Snell's Law

$\sin\theta_1 = \frac{x}{\ell_1}$

c_1 : speed of light in medium 1



c_2 : speed of light in medium 2

$\sin\theta_2 = \frac{a-x}{\ell_2}$

Fermat's principle: Light travels the path of fastest propagation

i.e. $T = \frac{\ell_1}{c_1} + \frac{\ell_2}{c_2}$ takes a minimum for the path chosen.

T: time the light takes to travel from source to observer

$h_1^2 + x^2 = \ell_1^2 \Rightarrow 2x = 2\ell_1 \frac{d\ell_1}{dx} \Rightarrow \frac{d\ell_1}{dx} = \frac{x}{\ell_1}$

$h_2^2 + (x-a)^2 = \ell_2^2 \Rightarrow 2(x-a) = 2\ell_2 \frac{d\ell_2}{dx}$

$\Downarrow$

$\frac{d\ell_2}{dx} = \frac{x-a}{\ell_2}$

$$\Rightarrow 0 = \frac{dT}{dx} = \frac{1}{c_1}\frac{d\ell_1}{dx} + \frac{1}{c_2}\frac{d\ell_2}{dx}$$

$$\Rightarrow 0 = \frac{1}{c_1}\sin\theta_1 + \frac{1}{c_2}(-\sin\theta_2)$$

$$\Rightarrow \boxed{\frac{c_1}{c_2} = \frac{\sin\theta_1}{\sin\theta_2}}$$ Snell's Law

Graph sketching

Goal: Find as many as possible qualitative features of the graph of a function using Calculus

1. Domain : $D(f)$
2. Intercepts (y-intercept, x-intercept if possible)

3. Horizontal asymptotes $\lim_{x \to \pm\infty} f(x)$

4. Vertical asymptotes $\lim_{x \to x_0} f(x)$ if x_0 is a boundary point of $D(f)$

5. First derivative: critical points, intervals where f is increasing / decreasing

6. Second derivative: points of inflection, intervals where f is concave up / down

Examples: ① $f(x) = x^4 - 6x^3 = x^3(x-6)$

1. $D(f) = \mathbb{R}$

2. $f(0) = 0$, so $x = y = 0$ is x and y-intercept

   $f(6) = 0$, so $x = 6$ is another x-intercept

3. $\lim_{x \to \pm\infty} f(x) = \infty$

4. no vertical asymptotes because $D(f) = \mathbb{R}$

5. $f'(x) = 4x^3 - 18x^2 = 2x^2(2x-9)$

   $\left.\begin{array}{l} f'(x) < 0 \quad \text{for} \quad 2x < 9 \quad \text{or} \quad x < \frac{9}{2} \\ f'(x) > 0 \quad \text{for} \quad x > \frac{9}{2} \end{array}\right\}$ f has a min. at $x = \frac{9}{2}$

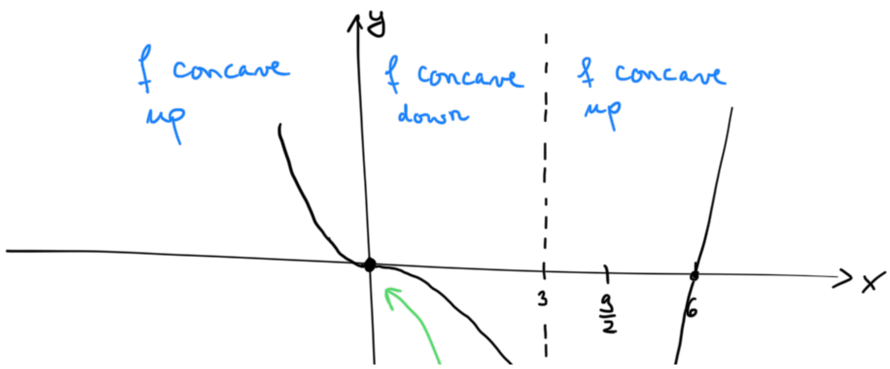
   Note: there is another critical point at $x = 0$, but $f'(x)$ does not change sign there.

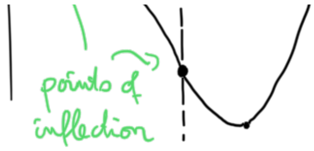
6. $f''(x) = 12x^2 - 36x = 12x(x-3)$

   If $x < 0$, then $f'' > 0$ $\Rightarrow$ f concave up "convex"

   $x \in (0,3)$, then $f'' < 0$ $\Rightarrow$ f concave down "concave"

   $x > 3$, then $f'' > 0$





② $f(x) = \sqrt[3]{\dfrac{x^2}{(x-6)^2}}$

1. $D(f) = \mathbb{R} \setminus \{6\}$

2. $f(0) = 0$ $(0,0)$ is x and y-intercept, no further x-intercepts

3. hor. asymptotes:

because $x \mapsto \sqrt[3]{x}$ is cont.

$$\lim_{x\to\infty} f(x) = \sqrt[3]{\lim_{x\to\infty} \frac{x^2}{(x-6)^2}} = 1 = \lim_{x\to-\infty} f(x)$$

$$\lim_{x\to\infty} \frac{1}{\left(1-\frac{6}{x}\right)^2}$$

$\frac{6}{x} \to 0$ as $x \to \infty$

$y=1$ is the hor. asymptote for $x \to \pm\infty$

4. vertical asymptotes

$$\lim_{x\to 6} f(x) = \infty$$ (both right and left-sided limit)

5. $f(x) = x^{\frac{2}{3}} (x-6)^{-\frac{2}{3}}$

$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}} (x-6)^{-\frac{2}{3}} + x^{\frac{2}{3}} \left(-\frac{2}{3}\right) (x-6)^{-\frac{5}{3}}$$

$$= \frac{2}{3} x^{-\frac{1}{3}} (x-6)^{-\frac{5}{3}} \left((x-6) - x\right) = -4 x^{-\frac{1}{3}} (x-6)^{-\frac{5}{3}}$$

$x^{-\frac{1}{3}} = \dfrac{1}{\sqrt[3]{x}}$

$D(f') = \mathbb{R} \setminus \{0, 6\}$

$x < 0 \Rightarrow f'(x) < 0$ $\quad f$ decreasing

$x \in (0,6) \Rightarrow f'(x) > 0$ $\quad f$ increasing

$x > 6 \Rightarrow f'(x) < 0$ $\quad f$ decreasing

$\Rightarrow$ at $x=0$, f has a local minimum

6. $f''(x) = -4\left(-\frac{1}{3} x^{-\frac{4}{3}} (x-6)^{-\frac{5}{3}} + x^{-\frac{1}{3}} \left(-\frac{5}{3}\right)(x-6)^{-\frac{8}{3}}\right)$

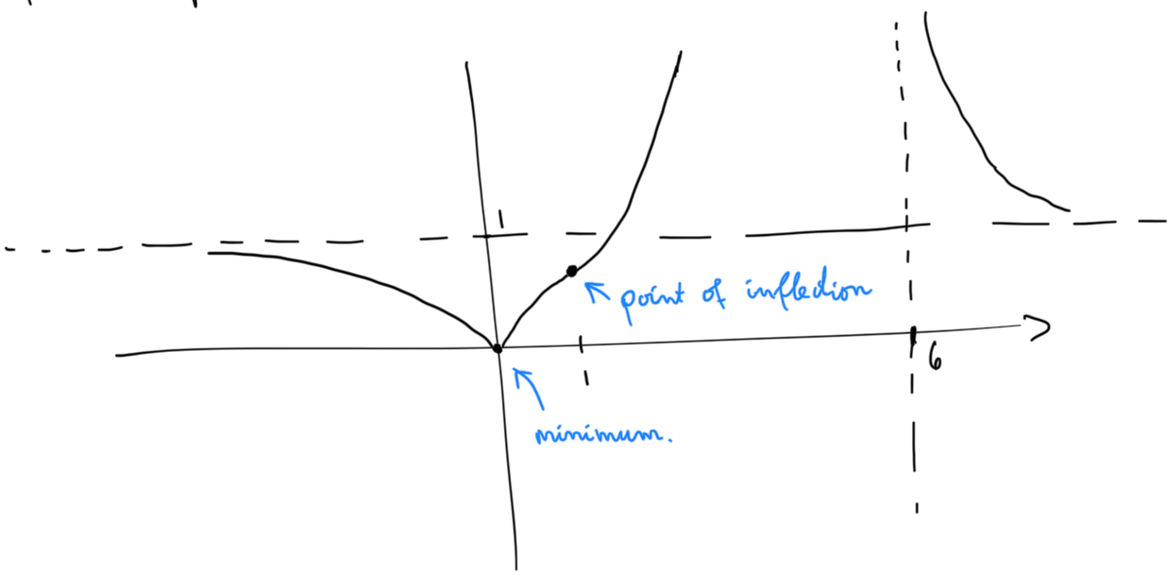
$x^{-\frac{4}{3}} x = x^{-\frac{4}{3}+1} = x^{-\frac{1}{3}}$

$$= \frac{4}{3} \underbrace{x^{-\frac{4}{3}} (x-6)^{-\frac{8}{3}}}_{\text{common denominator}} (\underbrace{x-6+5x}_{6(x-1)}) = 8 \underbrace{x^{-\frac{4}{3}}}_{\geq 0} \underbrace{(x-6)^{-\frac{8}{3}}}_{\geq 0} (x-1)$$

$\Rightarrow$ if $x<1$, $f''<0$, so f is concave down

$x>1$, $f''>0$ so f is concave up

f has a point of inflection at $x=1$



$f(g(x)) = h(x)$ $\qquad$ h cont., f cont.

- show that g does not need to be continuous

Simplest example: $f(x) = 0$

$\Rightarrow h(x) = 0$

whatever g is.

Note: situation is different if f invertible with cont. inverse:

$$\Rightarrow g(x) = \underbrace{f^{-1}}_{\text{cont.}}(\underbrace{h(x)}_{\text{cont.}}) \qquad \Rightarrow g \text{ cont.}$$

$f(x) = \sin\frac{1}{x}$